FINAL REPORT

NASA RESEARCH GRANT NGR-33-019-058

Contract Monitor: Felix L. Pitts
NASA Langley Research Center
Langley Station
Hampton, Virginia

SYNTHESIS OF DISTRIBUTED SYSTEMS

by

Gerald H. Cohen and Avinash R. Karnik

Period Covered: SEPT 1, 1966 - AUG 31, 1969

# THE UNIVERSITY OF ROCHESTER
# COLLEGE OF ENGINEERING AND APPLIED SCIENCE
# DEPARTMENT OF ELECTRICAL ENGINEERING
## ROCHESTER, NEW YORK

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# ABSTRACT

The work presented here demonstrates the use of direct
method of calculus of variations to solve circuit design problems.
In particular, a technique is developed for the synthesis and
design of a distributed parameter system guiding waves from one
point in space to another. The parameter distributions are
assumed to be unrestricted except for the upper and lower bounds
resulting from the imposition of practical realizability. The
problem is similar to the "sensitivity" problem encountered in
the optimal control of the systems. An improved version of the
First Order Gradient Technique is used to obtain the optimal
distributions of the parameters. The First Order Gradient
Technique is sensitive to the form of the arbitrary distributions
assumed at the start of the iterations. This technique has serious
convergence problems associated with it. The problem is particu-
larly severe and is encountered in the "singular" optimal control
problems. The algorithm devised here improves the First Order
Gradient Technique so that it becomes less sensitive to the
initial assumed distributions and virtually eliminates the
convergence problems generated because of the bounds on the para-
meter distributions.

A transmission line with distributed series rℓ and shunt c
is a particular case of the distributed parameter system. The
optimal design of a distributed rc filter for the feedback
circuit of an oscillator and the optimal design of a notched

filter employing a thin film circuit is a successful example of

the application of the Improved Gradient Technique. These

distributions have been obtained by the use of a Hybrid Computer.

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SYMBOLS

| | |
|---|---|
| $r(x)$ | Distributed series resistance |
| $c(x)$ | Distributed shunt capacitance |
| $\ell(x)$ | Distributed series inductance |
| $v(x,t)$ | Distributed line voltage |
| $i(x,t)$ | Distributed line current |
| $\alpha(x,\omega)$ | Amplitude of the voltage at frequency "$\omega$" |
| $\theta(x,\omega)$ | Phase angle of the voltage at frequency "$\omega$" |
| $L$ | Length of the line |
| $\Omega$ | Equation of constraint |
| $\phi$ | Criterion functional |
| $J$ | Optimal criterion functional |
| $y$ | $y^t=[y_1,y_2,y_3,y_4]=[V_1,V_2,I_1,I_2]$ |
| $\lambda$ | $\lambda^t=[\lambda_1,\lambda_2,\lambda_3,\lambda_4]$ |
| $u$ | $u^t=[u_1,u_2]=[r(x),c(x)]$ |
| $\lambda^\phi$ | Adjoint system of variables corresponding to $\phi$ correction |
| $\lambda^\Omega$ | Adjoint system of variables corresponding to $\Omega$ correction |
| $\delta u^\phi$ | $\phi$ component of the variation in control variable u |
| $\delta u^\Omega$ | $\Omega$ component of the variation in control variable u |
| $\Phi$ | Fundamental matrix of system equations |
| $\Psi$ | Fundamental matrix of adjoint equations |

$f$ $\qquad$ $f = \dfrac{d}{dx} y$

$<\lambda,f>$ $\qquad$ Inner product of $\lambda$ and $f$ vectors

$H$ $\qquad$ Hamiltonian, $H = <\lambda,f>$

$H^{\phi}$ $\qquad$ Hamiltonian corresponding to $\lambda^{\phi}$ system

$H^{\Omega}$ $\qquad$ Hamiltonian corresponding to $\lambda^{\Omega}$ system

$H_u$ $\qquad$ Partial differential operation

$$H_u = \left[ \frac{\partial H}{\partial u_1} , \frac{\partial H}{\partial u_2} \right]$$

$H_{yu}\delta y$ $\qquad$ $H_{yu}\delta y = \left[ \sum_i \frac{\partial}{\partial y_i} H \, \delta y_i \right]_u^t$

$\delta y H_{yu\lambda}\delta\lambda$ $\qquad$ $\delta y H_{yu\lambda}\delta\lambda = \left[ \sum_j \frac{\partial}{\partial y_j} \left( \sum_i \frac{\partial}{\partial \lambda_i} H \, \delta\lambda_i \right) \delta y_j \right]_u^t$

Sgn $\qquad$ Sign

L.H.S. $\qquad$ Left hand side

R.H.S. $\qquad$ Right hand side

AC $\qquad$ Analog Computer

DC $\qquad$ Digital Computer

ADC $\qquad$ Analog to Digital Converter

DAC $\qquad$ Digital to Analog Converter

Superscript t denotes transpose operation.

The limits for integration are x = 0 and x = L unless specified otherwise.

# CHAPTER 1

## 1.1 Introduction

Problems in the field of network synthesis with lumped para-
meters have been dealt with quite extensively in the existing
literature concerning circuit theory.

Systems with distributed parameters, specifically transmission
lines with a known distribution of resistance, capacitance, and
inductance, have been analyzed. However, the field of direct
synthesis of systems with distributed parameters is relatively
unexplored as seems evident from a few studies that have been
recently reported [1,2,3,4].

Most of the systems that transmit signals, such as a trans-
-mission line carrying electrical signal, a microwave channel carry-
ing electromagnetic wave, an acoustic horn transmitting sound waves,
or a blood vessel transmitting blood, are basically distributed
parameter systems. Since the advent of thin-film circuits the
synthesis problem has been introduced into the miniaturized circuitry.
Thin film circuits are replacing the lumped components due to the
requirement of

1). microminiaturization

and 2) modular construction.

A number of materials have been used for thin-film resistors
such as vacuum deposited nicrome, sputtered tantalum, vacuum

deposited metal oxides, etc. Thin-film capacitors are fabricated

by evaporating a high dielectric material on to a resistive path

and covering it with another layer of conducting film. The

dielectric layer may be formed by oxidization, e.g. silicon dioxide.

These various techniques enable us to realize nonuniform distribu-

tions. This can be achieved by controlling the physical dimensions

of the films. At very high frequencies it becomes necessary to

take into consideration the effect of the distributed inductance.

There exists a definite possibility of shaping the distributions

so as to optimize the performance of the system in which they are

used. Our aim is to develop a technique for optimal synthesis.

In the synthesis problem the viewpoint is maintained that the

energy is guided from one point to another in an integrated circuit

and the distributed parameters are determined as a function of

distance so as to achieve minimization of a cost functional. The

bounded but otherwise unrestricted distributions that we may have

to consider while handling the synthesis problem fall into the

category of nonuniform transmission lines. The problem we address

ourselves to is the optimal synthesis of nonuniform transmission

lines with bounded but otherwise unrestricted distributions.

Ekstrom [5] formulated a transfer matrix for such a trans-

mission line. However, for a general distribution the matrix can-

not be evaluated. Ekstrom [6] solved the system for an exponential

distribution of the parameters and obtained the transfer impedance

matrix. Castro and Happ [7] analyzed various configurations of rc thin-films with uniform distribution and derived transfer matrix for each case. The differential equations for the tapered transmission line have been solved in the frequency and the time domain for exponential and linear tapers [8,9]. Su [10] analyzed the rc line with $Sin^2$ and $Csc^2$ distributions.

The synthesis problem for the phase shift oscillator was considered by Edson [11]. Edson assumed an exponential taper and analyzed the problem in the frequency domain to obtain the gain of the line under the condition of $180^\circ$ phase shift. The technique used was to assume an exponential distribution of the type $r = Re^{\pm mx}$; $c = Ce^{\mp mx}$ with an unknown parameter "m". Then the differential equations were solved in the frequency domain in order to evaluate the performance for various values of "m" and the data achieved by this analysis was used for synthesis.

The problem in its true perspective was first taken up by Rohrer, Resh and Hoyt [1]. The general problem tackled in their paper is that of generating a distributed network from a given class which yields the best approximation to a desired time or frequency domain input output relationship. It should be noted that no restrictions have been put on the form of the distribution of the parameters except that the parameter values are bounded to take into account the problem of feasibility of building the circuit. The approach here has been basically different than the synthesis

techniques used in the past, in that their aim is to determine

the distribution of parameters so as to minimize in the integral

square sense the error between ideal output and the output of the

rc line. This technique resembles the optimal parameter control

for distributed systems. The above paper used the indirect method

of calculus of variations for minimization and ends up getting a

set of integral equations. An iterative technique has been used

[2] to solve these integral equations. Wohler, Kopp and Moyer [3]

used a direct method of calculus of variations to obtain solutions

to the impedance matching problem for a lossless transmission

line.

The aim of the present work is to develop a feasible technique

to obtain the distributions of parameters which optimize a

specified criterion, and to solve some hitherto unsolved practical

problems, such as a feedback circuit for the oscillator and the

notched filter.

1.2      Statement of the Problem .

For the integrated circuits interstages between active devices are typically series $r,\ell$ - shunt $c$, distributed networks as shown in Fig. 1.1.
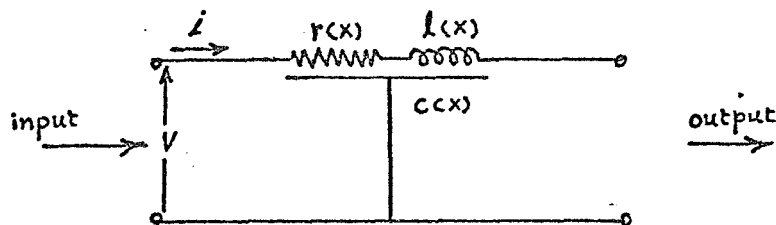


FIGURE 1.1

A DISTRIBUTED SERIES $r$, - SHUNT $c$ CIRCUIT

The frequency domain synthesis problem can be stated as follows.

Given the source voltage $\alpha(0,\omega)$ cos $(\omega t + \theta(0,\omega))$ over the frequency interval $(-\infty,\infty)$, find a series $r,\ell$ and shunt $c$ distributed network of length L which causes the output voltage $\alpha(L,\omega)$ cos $(\omega t + \theta(L,\omega))$ to be the best approximation to the desired output voltage $\alpha_d(L,\omega)$ cos $(\omega t + \theta(L,\omega))$ over the frequency interval $(-\infty,\infty)$. By letting $\theta(L,\omega)$ be free, we are requiring the match only in the magnitude of the frequency characteristics between the desired and actual output voltages. A further demand is that the element values be chosen from within some physically obtainable set, i.e. practical realizability dictates minimum and maximum values of $r(x)$ and $c(x)$.

Using the concept of the frequency response transfer function,

the absolute value of the system gain can be defined as,

$$\left| G_{(\omega)} \right| \;=\; \frac{\alpha(L,\omega)}{\alpha(0,\omega)} \quad . \qquad\qquad (1.2.1)$$

One can specify the desired frequency response $\left| G_d(\omega) \right|$ . Thus

we seek the distributions that give the best approximation to

the specified frequency response $\left| G_d(\omega) \right|$ for any input $v(0,\omega)$.

The system equations governing the relationship between the

voltage and current in Fig. 1.2 are partial differential equa-

tions in time and space.
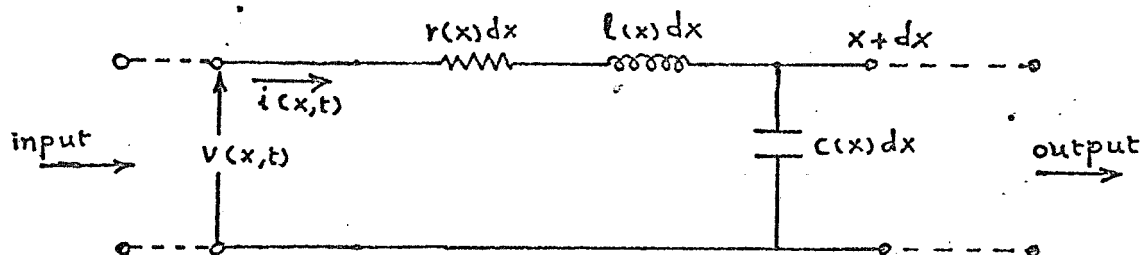


FIGURE 1.2

A SECTION OF A DISCRETIZED SERIES $r$, - SHUNT c CIRCUIT

$$\frac{\partial}{\partial x}[v(x,t)] + r(x)i(x,t) + \ell(x)\frac{\partial}{\partial t}[i(x,t)] \;=\; 0 \;,$$

$$\qquad\qquad (1.2.2)$$

$$\frac{\partial}{\partial x}[i(x,t)] + c(x)\frac{\partial}{\partial t}[v(x,t)] \;=\; 0 \;. \qquad (1.2.3)$$

The driving function $v(0,t)$ is assumed to be a co-sinusoidal

input at a frequency $\omega$. Linearity and time invariance of para-

meters $r,c$ and $\ell$ assure the presence of only one frequency "$\omega$".

Thus we can assume[1] a steady state solution of the form

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[1]Schelkunoff [12] shows that the partial differential equa-
tion for nonuniform transmission lines are separable in time and
space.

$$v(x,t) = \alpha(x,\omega) \cos(wt+\theta(x,\omega)) = V_1(x,\omega) \cos \omega t + V_2(x,\omega) \sin \omega t,$$

$$i(x,t) = I_1(x,\omega) \cos \omega t + I_2(x,\omega) \sin \omega t,$$

where $\theta(x,\omega) = \tan \dfrac{V_2(x,\omega)}{V_1(x,\omega)}$ specifies the phase angle of the

voltage as a function of x, and $\alpha(x,\omega) = (V_1^2(x,\omega) + V_2^2(x,\omega))^{1/2}$

gives the amplitude of the voltage along the line. Substituting

this solution into the equation (1.2.2) and (1.2.3) we obtain time

independent state equations

$$\frac{d}{dx}V_1(x,\omega) = -r(x)I_1(x,\omega) - \omega\ell(x)I_2(x,\omega) = f_1 ,$$

$$\frac{d}{dx}V_2(x,\omega) = -r(x)I_2(x,\omega) + \omega\ell(x)I_1(x,\omega) = f_2 ,$$

$$\frac{d}{dx}I_1(x,\omega) = -\omega c(x)V_2(x,\omega) = f_3 ,$$

$$\frac{d}{dx}I_2(x,\omega) = \omega c(x)V_1(x,\omega) = f_4 , \qquad (1.2.4)$$

The presence of "$\omega$" as an independent variable for $V_1$, $V_2$, $I_1$, $I_2$

identifies these system variables as trajectories for an input at

frequency "$\omega$".

Since the behavior of the ratio of the output and the input

voltages is of interest, one can specify the voltage at either

end of the line. Without any loss of generality we can specify,

$$V_1(L,\omega) = a ,$$
$$V_2(L,\omega) = 0 . \qquad (1.2.5)$$

where "a" is a nonzero constant. We assume an open circuit at the

output end implying thereby

$$I_1(L,\omega) = 0,$$

$$I_2(L,\omega) = 0. \tag{1.2.6}$$

For any known load admittance

$$Y_L = Y_r + jY_m,$$

where $Y_r$ is the real part and $Y_m$ is the imaginary part of the
complex admittance $Y_L$, the terminal current is

$$I_1(L,\omega) = aY_r,$$

$$I_2(L,\omega) = aY_m.$$

The desired frequency response may be low pass, band pass, or
may have any general form as in Fig. 1.3.
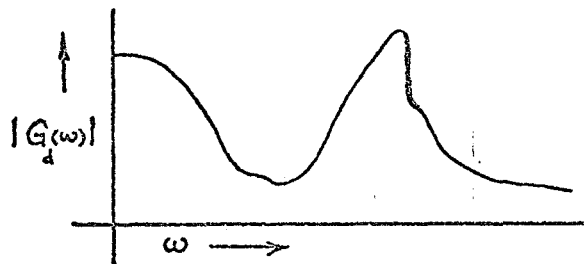


FIGURE 1.3

AN ARBITRARY 'DESIRED' FREQUENCY RESPONSE CHARACTERISTICS.
In order to obtain an approximation to such a response one can
formulate a quadratic functional

$$\phi = \int_{-\infty}^{\infty} ||G_d(\omega)|^2 - |G(\omega)|^2| d\omega . \tag{1.2.7}$$

Noting from (1.2.5) that $v(L,\omega)$ has been completely specified,
$|G(\omega)|$, as defined in (1.2.1) becomes a function of $v(0,\omega)$ alone.

Define

$$F(V_1(0,\omega),\ V_2(0,\omega)) \ = \ |\,|G_d(\omega)|^2 - |G(\omega)|^2\,|.$$

We can rewrite the criterion function

$$\phi \ = \ \int_{-\infty}^{\infty} F[V_1(0,\omega),\ V_2(0,\omega)]d\omega \quad . \qquad (1.2.8)$$

Our aim is to find out $r(x)$ and $c(x)$ such that $\phi$ is minimized. The inductance $\ell(x)$ is assumed to be a non-controllable parameter. Its value is decided by the geometry of the circuit.

Defining the state variable $y$ as

$$y \ = \ \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} , \qquad (1.2.9)$$

we can restate the problem as:

$$\text{Given} \ \frac{d}{dx} y \ = \ Ay \ = \ f(u,y) \ , \qquad (1.2.10)$$

where the matrix

$$A \ = \ \begin{bmatrix} 0 & 0 & -r(x) & -\omega\ell(x) \\ 0 & 0 & \omega\ell(x) & -r(x) \\ 0 & -\omega c(x) & 0 & 0 \\ \omega c(x) & 0 & 0 & 0 \end{bmatrix} , $$

$$(1.2.10a)$$

with the boundary conditions

$$y^t(x=L) = y_L^t = [a,0,0,0] \quad , \qquad (1.2.11)$$

find $u=u_{optimal}$ such that

$$J = \min_u \phi \quad , \qquad (1.2.12)$$

where[2]

$$\phi = \int_{-\infty}^{\infty} F(y_1(0,\omega), y_2(0,\omega))d\omega \qquad (1.2.13)$$

and the control vector

$$u = \begin{bmatrix} r(x) \\ c(x) \end{bmatrix} \quad ,$$

subject to the inequality constraints

$$u_{min} \leq u(x) \leq u_{max} \quad , \qquad (1.2.14)$$

implying

$$r_{min} \leq r(x) \leq r_{max} \quad ,$$

$$c_{min} \leq c(x) \leq c_{max} \quad . \qquad (1.2.15)$$

The control vector $u$ enters the system equations (1.2.10) in a linear manner and is totally absent from the criterion function (1.2.13). As a result we have what is termed in optimal control theory "a singular problem" [13]. The optimal control satisfying (1.2.12) may be

---

[2]$F$ may be a function of $y(0,\omega)$ in general.

(i)    bang bang type, meaning thereby that u may be $u_M$ or $u_m$ with step transition or switching (Fig. 1.4) or

(ii)   limiting control with singular switching curves (Fig. 1.5) or

(iii)  staying entirely within the open region and not reaching the boundaries.

The mathematical tools available for tackling this problem can be categorized as follows:

(i)    The first group consists of the approaches based on obtaining a set of necessary conditions for optimality. These may be the Euler Lagrange differential equations, with mixed boundary conditions obtained from the transversality relationship; or a set of first order state and adjoint differential equations with mixed boundary conditions derived from the Maximum Principle, or a Hamilton-Jacobi type partial differential equation derived by using Bellman's approach of Dynamic Programming. If one can somehow find the analytic solution to these set of equations, one may have found the control that optimizes the criterion functional (very rarely does one utilize the criterion for sufficiency).

In the numerical analysis aimed at obtaining the approximate solutions to the set of necessary conditions, some kind of iterative procedure is used. However, the successive iterations are not geared to obtaining uniformly decreasing values of $(J-\phi)$. For example, in solving a two point boundary value problem the successive iterations may monotonically reduce the difference
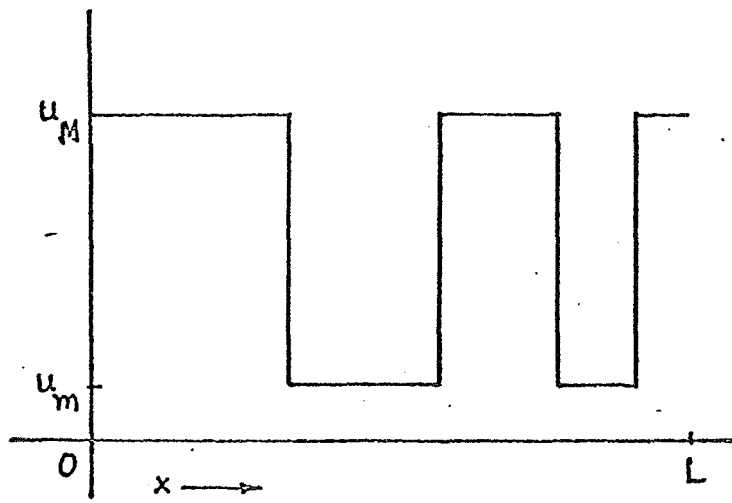
FIGURE 1.4

A BANG BANG TYPE CONTROL
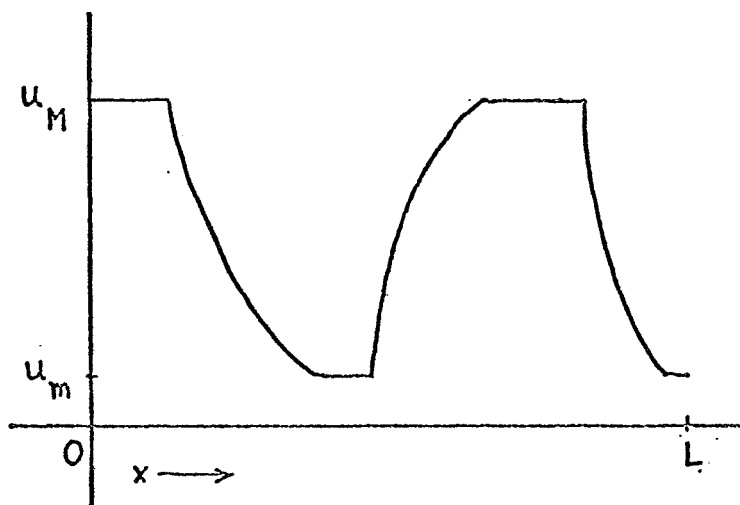


FIGURE 1.5

A LIMITING CONTROL WITH SWITCHING CURVES

between the specified boundary values and the ones that are

obtained during iterations. But this does not necessarily imply

that the approximate solutions generated during the iterations

will monotonically improve the criterion functional. Thus the

approximate solutions to the set of necessary conditions may not

be approximately optimal in the sense that $(J-\phi)$ may not be small

enough to call the trajectories approximately optimal.

(ii) The alternate approach is to use one of the direct

methods of calculus of variations and seek gradual improvement in

the criterion functional. The method is known as "Gradient

Technique" [14], "Relaxation Method", or "Hill Climbing Technique".

It is like climbing a hill in foggy weather, by estimating the

terrain in the neighborhood of the present position and proceeding

in the steepest upward direction in order to reach the "highest"

altitude. Every step improves the position. To be more specific,

one assumes an arbitrary non-optimal u and then seeks a stepwise

improvement in the direction of the optimum. The new values of

u generated at every step of iteration result in an improvement

in the value of the criterion functional. The process hopefully

converges to the optimum.

In order to take an iterative step in the proper direction

one must have a reliable estimate of the behavior of the system

in the neighborhood of the assumed arbitrary control. Specifically,

we seek to obtain a functional relationship between variations in

the control vector u and the resulting variation in the criterion
functional. This yields a self-sufficient iterative procedure.
We will use this gradient method in some of its various forms
developed in the sequel along with the algorithms that allow us
to take advantage of the Hybrid Computational Technique.

1.3      Gradient Method

The basic equations used in the algorithms for the gradient

method can be derived as follows.  Referring to the set of

equations (1.2.10) we have

$$\frac{d}{dx} y_i(x,\omega) = f_i(y,\omega,u) , \qquad i=1,\dots 4. \qquad (1.3.1)$$

For using the gradient technique one has to consider a small

perturbation in the control variable u.  With new control as

u+δu and resulting trajectories as y+δy, the resulting first order

variational equations are given by

$$\frac{d}{dx}(\delta y_i) = \sum_{j=1}^{4} \frac{\partial f_i}{\partial y_j} \delta y_j + \sum_{k=1}^{2} \frac{\partial f_i}{\partial u_k} \delta u_k . \qquad (1.3.2)$$

In matrix form (1.3.2) can be restated as

$$\frac{d}{dx}\delta y = A\delta y + A_u y\delta u , \qquad (1.3.2a)$$

where $A_u$ denotes the partial differentiation of "A" matrix with

reference$_j$ to the subscripted variable u.  With A defined as in

(1.2.10a)

$$A_u y\delta u = \begin{bmatrix} -y_3 & 0 \\ -y_4 & 0 \\ 0 & -\omega y_2 \\ 0 & \omega y_1 \end{bmatrix} \begin{bmatrix} \delta r \\ \delta c \end{bmatrix}$$

Now one can define a set of adjoint variables by the differential equations

$$\frac{d}{dx} \lambda_i(x,\omega) = -\sum_j \frac{\partial f_j}{\partial y_i} \lambda_j(x,\omega) \quad . \tag{1.3.3}$$

The system equations (1.2.10) are linear in y. Under such circumstances the adjoint equations will always be reduced to a form

$$\frac{d}{dx} \lambda(x,\omega) = -A^t(u(x),\omega)\lambda(x,\omega) \quad . \tag{1.3.4}$$

Before illustrating the function of the adjoint variables, we can discuss their form. From (1.2.10) and (1.3.4) one can derive the relationship

$$\frac{d}{dx}(\lambda^t y) = 0 \quad ; \tag{1.3.5}$$

i.e. the inner product of $\lambda$ and y remains constant for all x. This implies,

$$\lambda^t(0)y(0) = \lambda^t(L)y(L) \quad . \tag{1.3.6}$$

In terms of fundamental matrices [15] $\phi(x)$ and $\psi(x)$, where

$$y(x) = \phi(x)y(0) \quad , \tag{1.3.7}$$

$$\lambda(x) = \psi(x)\lambda(0) \quad , \tag{1.3.8}$$

the relationship (1.3.6) yields

$$\psi(x) = \phi^{-1}(x) \quad . \tag{1.3.9}$$

Hence, if the solutions to (1.2.10) are known in terms of the

fundamental matrix $\Phi$, the solutions for adjoint equations can be
obtained as

$$\lambda(x) = \Phi^{-1}(x)\lambda(0)$$

without solving (1.3.4).

Also for certain forms of matrix A there exists a simple
linear transformation of the type

$$\gamma(x) = B\lambda(x) \qquad (1.3.10)$$

where B is a nonsingular constant matrix, such that, with this
transformation the adjoint equations

$$\frac{d}{dx}\lambda = -A^t\lambda$$

become

$$\frac{d}{dx}\gamma = -A\gamma \quad .$$

With the change of variable as

$$z = L - x \qquad (1.3.11)$$

the equations reduce to

$$\frac{d}{dz}\gamma(z) = A(z)\gamma(z) \quad . \qquad (1.3.12)$$

This equation has a form identical to (1.2.10). Thus the trans-
formed adjoint variables $\gamma(z)$ are solutions of the system equa-
tions with the reversal of the space variable. The "A" matrix as
defined by (1.2.10) possesses the above properties. We will
consider the corresponding B matrix and the significance of the

property mentioned about later, when we will get to the stage

of obtaining the numerical solutions to the system and the

adjoint equations.

Multiplying equation (1.3.1) by $\lambda_i$, (1.3.3) by $\delta y_i$, adding

them together and performing summation over i, we get

$$\frac{d}{dx} [\sum_i \lambda_i \delta y_i] = \sum_i \sum_k \lambda_i \frac{\partial f_i}{\partial y_k} \delta u_k \quad . \qquad (1.3.13)$$

Define

$$H = \langle \lambda, f \rangle = \sum_i \lambda_i f_i \quad .$$

Then, integrating (1.3.13) from x=0 to x=L,

$$[\sum_i \lambda_i y_i]_0^L = \int_0^L H_u \delta u \, dx \quad . \qquad (1.3.14)$$

Let us consider a general case where mixed boundary condi-

tions have been satisfied. Divide y into two sets.

$$y^t = [y_{in}^t, y_{f\ell}^t] \quad ,$$

such that

$$y_{in}^t = [y_1, \ldots, y_k] \quad ,$$

and

$$y_{f\ell}^t = [y_{k+1}, \ldots, y_n] \quad . \qquad (1.3.15)$$

The mixed boundary conditions are

$$y_{in}(x=0, \omega) = y_{in}(\omega) \quad ,$$

$$y_{f\ell}(x=L, \omega) = y_{f\ell}(\omega) \quad . \qquad (1.3.16)$$

Then we can consider a general criterion function as

$$\phi = \int_{-\infty}^{\infty} F(y_{in}(x=L,\omega), y_{f\ell}(x=0,\omega))d\omega \qquad (1.3.17)$$

Since $y_{in}(0,\omega)$ and $y_{f\ell}(L,\omega)$ have been specified,

$$\delta y_{in}(0,\omega) = 0$$

and

$$y_{f\ell}(L,\omega) = 0 \quad .$$

So far we have not defined the boundary conditions for the adjoint system. Let us define

$$\lambda_{in}^{t} = [\lambda_1, \ldots \lambda_k] \quad , \qquad (1.3.18)$$

$$\lambda_{f\ell}^{t} = [\lambda_{k+1}, \ldots \lambda_n] \quad . \qquad (1.3.19)$$

In order to obtain the functional relationship between the variation in criterion function and the control variable, define the boundary conditions as

$$\lambda_{in}(x=L,\omega) = F_{y_{in}}^{t}(\omega) \qquad (1.3.20)$$

and

$$\lambda_{f\ell}(x=0,\omega) = F_{y_{f\ell}}^{t}(\omega) \qquad (1.3.21)$$

where $F_{y_{in}}(\omega)$ and $F_{y_{f\ell}}(\omega)$ are the partial differentials of F with respect to $y_{in}$ and $y_{f\ell}$ .

Thus the left hand side of (1.3.14) becomes.

$$[\sum_i \lambda_i \delta y_i]_0^L = [<\lambda_{in} \delta y_{in}> + <\lambda_{f\ell} \delta y_{f\ell}>]_0^L .$$

By using (1.3.15), (1.3.20) and (1.3.21), it reduces to

$$<F_{y_{in}} , \delta y_{in}> + <F_{y_{f\ell}} , \delta y_{f\ell}> = dF .$$

Hence (1.3.14) reduces to

$$dF(\omega) = \int H_u(x,\omega)\delta u(x)dx . \qquad (1.3.22)$$

Integrating (1.3.22) over "$\omega$" and using the relationship (1.3.17) we obtain

$$d\Phi = \int_{-\infty}^{\infty} \int_0^L H_u \delta u dx d\omega . \qquad (1.3.23)$$

The above expression gives a functional relationship between the variation in control and change in criterion function in response to it. Noting that $\delta u$ is a function of x one realizes that an unlimited number of solutions for $\delta u$ can be obtained from (1.3.23) for a specified value of $d\Phi$. Hence we stipulate an arbitrary criterion function

$$\Phi_s = \frac{1}{2} \int \delta u^t W \delta u dx \qquad (1.3.24)$$

which has to be minimized while satisfying (1.3.23). The matrix $W(x)$ is a square weighting factor matrix. The choice of $W(x)$ is quite arbitrary except that it has to be positive for all x. This is required in order to satisfy the strengthened Legendre condition for the minimum of $\Phi_s$. This

(i) eliminates the singular problem since the criterion is quadratic in control, and

(ii) keeps the variation $\delta u$ to a minimum (in a Euclidian norm sense). This is desirable since the derivations are based on a small perturbation.

We have to find $\delta u$ that minimizes the composite criterion function

$$\psi = \frac{1}{2} \int \delta u^t W \delta u dx + \nu[d\phi - \int_{-\infty}^{\infty} \int H_u \delta u dx d\omega] \qquad (1.3.25)$$

where $\nu$ is an undetermined Lagrange multiplier, to be chosen so as to satisfy (1.3.23). Euler Lagrange equations give

$$\delta u = \nu \int_{-\infty}^{\infty} W^{-1} H_u d\omega \qquad (1.3.26)$$

Substituting this result back in (1.3.23) we obtain the expression for the Lagrange multiplier $\nu$,

$$\nu = d\phi / \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{L} H_u(x,\omega) W^{-1}(x) H_u(x,\gamma) \, dx d\omega d\gamma \qquad .$$

We finally have the expression we were seeking for the optimal $\delta u$.

$$\delta u = [d\phi / \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{L} H_u(x,\omega) W^{-1}(x) H_u(x,\gamma) dx d\omega d\gamma]$$

$$\int_{-\infty}^{\infty} W^{-1}(x) H_u(x,\omega) d\omega \qquad . \qquad (1.3.27)$$

The iteration algorithm is fairly straightforward and proceeds as follows.

(1) Assume a nominal control u. ' Solve system equations (1.3.1) with boundary conditions (1.3.16).

(2) Solve the adjoint equations (1.3.3) with boundary conditions (1.3.20) and (1.3.21) to obtain $\lambda$.

(3) Evaluate $\delta u$ from (1.3.27) for a stipulated value of [3]$d\phi$ and an assumed value of [4]$W^{-1}$.

(4) Add $\delta u$ to u and obtain revised estimate for control as u+$\delta u$. The inequality constraints on u are taken into account by truncating (u+$\delta u$) at $u_M$ or $u_m$ wherever it crosses the bounds. The validity of truncation can be proven (refer Appendix B). Now branch back to the start of the loop for the next iteration cycle.

The procedure is repeated till the improvement $d\phi$ corresponding to a "reasonably" small $\delta u$ drops down considerably. By this time the system hopefully converges to at least a local optimum. The weighting matrix W(x) is found to be a critical factor in influencing the convergence. We postpone the discussion of convergence till we face that problem in the next chapter.

To obtain a better understanding of the nature of the variation $\delta u$, let us define

---

[3]$d\phi$ is chosen so as to obtain convergence of $\phi$ to an optimum value.

[4]The choice of $W^{-1}$ is based on the knowledge of the system. The unity matrix could be chosen as a first estimate of $W^{-1}$. Section 2.4 deals extensively with the choice of $W^{-1}$.

$$\delta u_s(x) = \int_{-\infty}^{\infty} W^{-1}(x) H_u(x,\omega) d\omega \quad . \tag{1.3.28}$$

Hence (1.3.26) can be rewritten as

$$\delta u(x) = \nu \delta u_s(x) \quad . \tag{1.3.29}$$

There are two aspects of the form of $\delta u$ which we can influence:

1. The shape of the variation $\delta u_s(x)$, as influenced by the arbitrary shaping matrix $W^{-1}(x)$; and

2. The amount of variation, or the step size $\nu$, which is constant for all $x$. The arbitrariness in the choice of $d\phi$ influences $\nu$.

Let us consider an analogy from the field of ordinary calculus. Let the criterion function $\phi$ which has to be minimized be a function of two independent variables $x_1$ and $x_2$

$$\Phi = \Phi(x_1, x_2) \quad . \tag{1.3.30}$$

Hence the first order variational equation is

$$d\Phi = \Phi_{x_1} \delta x_1 + \Phi_{x_2} \delta x_2 \tag{1.3.31}$$

where $\Phi_{x_1}$ and $\Phi_{x_2}$ are partial differentials of $\Phi$ with respect to $x_1$ and $x_2$ respectively.

For a given $d\Phi$ we can find non-unique values of $\delta x_1$ and $\delta x_2$. An additional constraint that removes the non-uniqueness is obtained when we seek a variation $(\delta x_1, \delta x_2)$ that

(i) minimizes

$$\frac{1}{2} \parallel \delta x \parallel^2 = \frac{1}{2} (\delta x_1^2 + \delta x_2^2) \qquad (1.3.32)$$

and (ii) satisfies (1.3.30).

The composite criterion function for this accessory minimization problem can be written down (refer (1.3.25)).

$$\psi = \frac{1}{2} (\delta x_1^2 + \delta x_2^2) + \mu[d\phi - (\phi_{x_1} \delta x_1 + \phi_{x_2} \delta x_2)] \ .$$

The conditions for stationarity of $\psi$ yield

$$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = \mu \begin{bmatrix} \phi_{x_1} \\ \phi_{x_2} \end{bmatrix} \qquad (1.3.33)$$

Substituting this back into (1.3.31) we get

$$\mu = d\phi / |Grad\phi|^2 \ .$$

In (1.3.33) the gradient $\begin{bmatrix} \phi_{x_1} \\ \phi_{x_2} \end{bmatrix}$ gives the direction in $(x_1, x_2)$

space and $\mu$ is the step size.

Referring back to the problem of the calculus of variations if we let $W^{-1}$ equal identity matrix, $\delta u_s$ in (1.3.29) is the "gradient" of $\phi$ at any x. We can call this function a "shape" of the variation that specifies the "direction" in y space at all x. The constant $\nu$ in (1.3.29) is comparable to the step size $\nu$

in the above example. Thus, the separation of the variation in control $\delta u$ as a "shape factor" $\delta u_s$ and a step size $\nu$ is comparable to the "gradient" and a step size.

So far we have considered a general case and have tried to bring out certain points and describe certain properties that will be made use of when we try to handle particular cases in the following chapters. The numerical analysis will be done with the help of a Hybrid Computer unit. The problem will be discretized in frequency.

As a first stage we will consider the system operation at only one frequency. This is the synthesis of a feedback circuit for the oscillator. Then we will consider the design of a notched filter where one has to shape the frequency response of the filter.

CHAPTER 2

## 2.1    Phase-Shift Oscillator Feedback Network

The design of a feedback network for a phase shift oscilla-
tor has been a topic of a number of studies. As shown in a very
basic block diagram (Fig. 2.1), the frequency sensitive feedback
circuit has to provide a proper
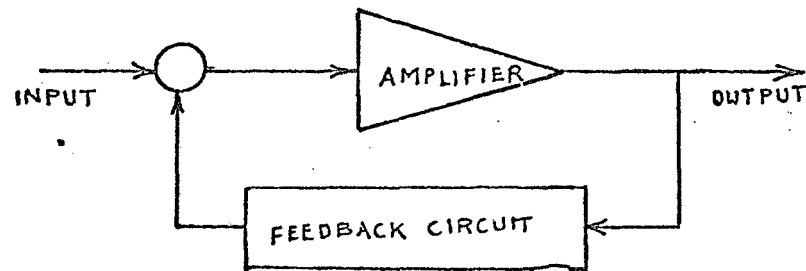


FIGURE 2.1

A BLOCK DIAGRAM FOR THE PHASE SHIFT OSCILLATOR

gain and phase shift relationship so that the system may oscillate.
Assuming that the amplifier has a phase shift of 180 degrees, the
feedback circuit has to provide an additional 180 degree phase
shift in order to get the conditions for oscillation. One other
usual requirement of the feedback circuit is that the attenuation
during the transmission of the signal should be minimum, since
the total gain around the loop should be unity. This lowers the
gain requirements of the amplifier.

A three section lumped parameter -- series r, shunt c --
network, where all the sections have finite nonzero resistance
and capacitance values, gives [16] an attenuation of 29 for 180

degree phase shift. When each section of such n section network

is completely isolated, i.e. it does not load the previous

section, the gain corresponding to 180 degree phase shift is

given [17] by

$$\text{gain} = \sec^n \frac{\pi}{n} \quad .$$

The table 1 shows the values of attenuation for circuits with n

equal to one to eight. Johnson [18] has shown that the circuit

in Fig. 2.2 gives, in the limit, an attenuation of 8 as K tends
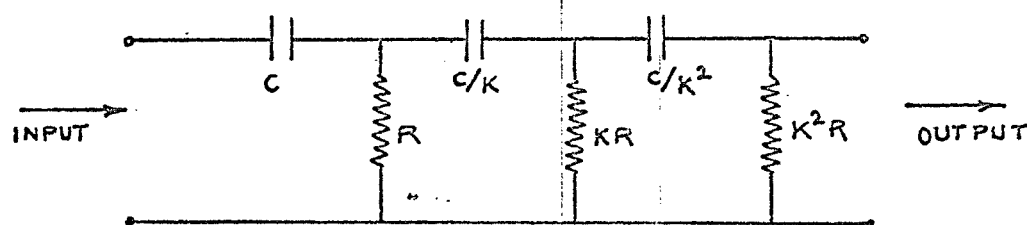


FIGURE 2.2

A THREE SECTION FEEDBACK CIRCUIT USED BY JOHNSON [18]

to infinity. He also showed that a uniformly distributed series

r, parallel c network would produce an attenuation of 11.6.

Increasing the number of sections in the lumped parameter circuit

helps to reduce the attenuation. A limiting case is obviously a

distributed rc transmission line. At very high frequencies the

distributed series inductance begins to come into picture. Edson

[11] states that, "unfortunately, the analysis of multiple-

section lumped networks is exceedingly complicated and tedious...".

It is found that useful inferences may be drawn from the limiting

case in which the number of sections becomes infinite and the

TABLE 1

THE ATTENUATION OF SERIES r SHUNT c, n SECTION
NETWORK WITH 180 DEGREE PHASE SHIFT.
ALL THE SECTIONS ARE IDENTICAL
AND ISOLATED.

| No. of Sections n | Attenuation Sec.$^n\frac{\pi}{n}$ |
|---|---|
| 1 | $\infty$ |
| 2 | $\infty$ |
| 3 | 8 |
| 4 | 4 |
| 5 | 2.9 |
| 6 | 2.37 |
| 7 | 2.108 |
| 8 | 1.884 |

network becomes a smoothly tapered transmission line". (Fig. 1.1)
He assumed an exponential variation of the parameters correspond-
ing to

$$r(x) = Re^{\pm 2kx} ,$$

$$c(x) = Ce^{\mp 2kx} ,$$

$$\ell(x) = 0 .$$

Edson obtained curves for attenuation at 180 degree phase shift
as a function of parameters R,C, taper k, and line length L. It
can be easily shown that as k approaches infinity the attenuation
approaches unity.

The approach we will use in the sequel is to keep the form
of the distributions completely free except for the upper and
lower bounds on r and c resulting from the physical constraints
of realizability and try to obtain the distributions of para-
meters r(x) and c(x) which optimize the specified criterion,
viz. minimum attenuation at 180 degree phase shift.

## 2.2    Mathematical Formulation of the Oscillator Problem

The general statement of the problem is as follows:

For a distributed -- series r and $\ell$, shunt c -- feedback circuit find the distributions r(x) and c(x) [with reference to Fig. 1.1] such that at a given frequency,

(i)  There exists a desired phase shift between the input and output, and

(ii)  the attenuation is minimum.

The distributed inductance $\ell(x)$ is assumed to be a non-controllable quantity.

The system equations governing the relationship between the voltage and current in Fig. 1.1 are partial differential equations in time and space as given in (1.2.2) and (1.2.3). The driving function $v_{in}(0,t)$ is a cosinusoidal input at frequency "$\omega$". We can assume a steady state solution of the form

$$v(x,t) = \alpha(x) \cos (\omega t + \theta) = V_1(x) \cos \omega t + V_2(x) \sin \omega t$$

$$(2.2.1)$$

$$i(x,t) \qquad\qquad = I_1(x) \cos \omega t + I_2(x) \sin \omega t$$

$$(2.2.2)$$

where $\theta(x) = \tan^{-1} \dfrac{V_2(x)}{V_1(x)}$ specifies the phase angle of the

voltage as a function of x and $\alpha(x) = (V_1^2(x) + V_2^2(x))^{1/2}$ is the amplitude of the voltage along the line. Substituting the solutions into (1.2.2) and (1.2.3) we get the set of equations

(1.2.4) for fixed $\omega$.

At this stage we make two trivial assumptions.

(i) The output impedance of the amplifier [source impedance at the input of the line] is zero.

(ii) The input impedance of the amplifier [load impedance on the line] is infinite.

Without any loss of generality the input conditions of the line could be specified as

$$V_1(0) = a \ , \qquad a > 0 \ ,$$

$$V_2(0) = 0 \ . \tag{2.2.3}$$

The open circuit at the output end of the line implies

$$I_1(L) = I_2(L) = 0 \ . \tag{2.2.4}$$

The two conditions given above get slightly modified for a non-zero amplifier output impedance and finite amplifier input impedance.

The 180 degree phase shift requirement is translated as

$$V_2(L) = 0 \ ,$$

$$V_1(L) < 0 \ . \tag{2.2.5}$$

Equation (2.2.5) assures a phase shift of $\pi$, $3\pi$, $5\pi$, .....

Since the minimum attenuation is same as the maximum gain needed to maximize $\phi$,

$$\phi = \frac{\alpha(L)}{\alpha(0)} \qquad (2.2.6)$$

with $\alpha(x)$ as defined in (2.2.1). Since $V_2(0) = V_2(L) = 0$ and $V_1(0) = a$, the criterion becomes

$$\max \phi = \max \left| V_1(L) \right| . \qquad (2.2.7)$$

The general forms of the solution will be as shown in Fig. 2.3.
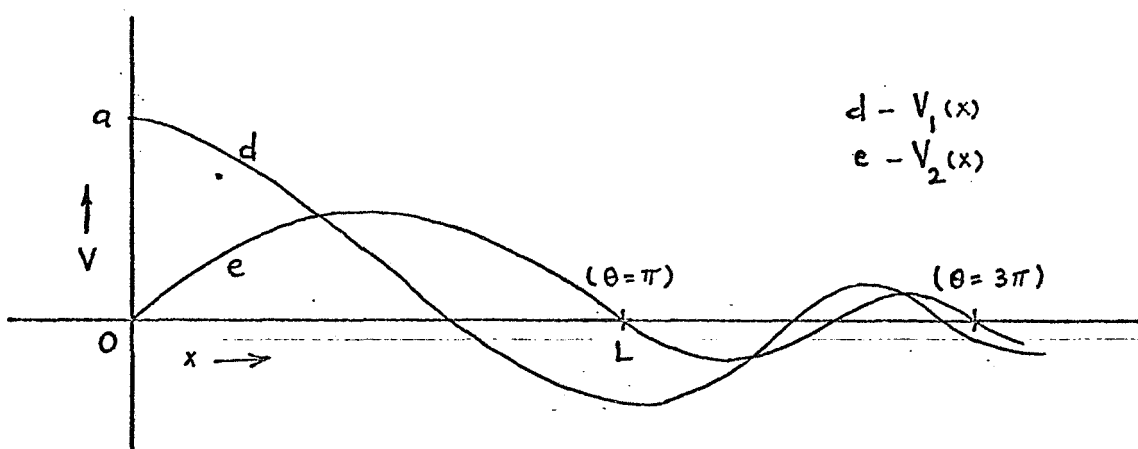


FIGURE 2.3

A GENERAL FORM OF THE IN-PHASE AND OUT-OF-PHASE COMPONENTS
OF THE VOLTAGE WAVE ALONG THE TRANSMISSION LINE

The attenuation increases as the signal travels along the line. Thus there is no possibility of the attenuation at a phase shift of $3\pi$, $5\pi$, ... being smaller than that at $\pi$. We can safely restrict our considerations to the phase shift of $\pi$, or the first zero of $V_2(x)$.

The boundary conditions (2.2.3) and (2.2.4) require solving a two point boundary value problem, since the voltage is specified at one end and the current at the other end. It is possible to avoid mixed boundary conditions by specifying the voltage at x=L.

If the conditions are specified as $V_1(L)=a$,
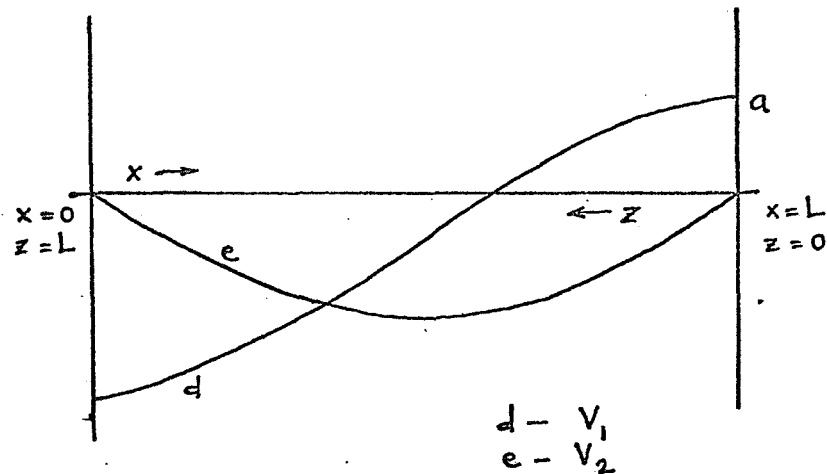


FIGURE 2.4

THE IN-PHASE AND OUT-OF-PHASE COMPONENTS OF THE VOLTAGE
WAVE FOR 180 DEGREE PHASE SHIFT

$V_2(L)=0$, from (2.2.6) it is apparent that with

$$V_2(0) = I_1(L) = I_2(L) = 0 ,$$

$$\max \phi = \min |V_1(0)| .$$

Fig. 2.4 shows a general form of the solution. For $a > 0$,
$V_1(0) < 0$. This implies that

$$\max \phi = \max V_1(0) \tag{2.2.8}$$

Now we can define the problem using control system terminology.
Define the four-vector

$$y^t = [V_1, V_2, I_1, I_2]$$

The system equations are given by

$$\frac{d}{dx} y(x) = A(u(x))y(x) \tag{2.2.9}$$

The matrix $A(u(x))$ is defined by (1.2.10a) and $u(x)$ is a two-vector defined by

$$u^t(x) = [r(x), c(x)] \; . \tag{2.2.10}$$

The inductance $\ell(x)$ is assumed to be a non-controllable parameter. The endpoint boundary conditions are

$$y^t(x=L) = [a,0,0,0] \tag{2.2.11}$$

with L fixed; and the rigid constraint $\Omega$ is given by

$$\Omega[\dot{y}(x=0)] = y_2(0) = 0 \; . \tag{2.2.12}$$

Our task is to obtain $r$ and $c$ distributions that maximize $\phi$, where

$$\phi = y_1(0) \tag{2.2.13}$$

We also assume that the limitations in fabrication require that the values of resistance and capacitance per unit length be within finite upper and lower bounds. This gives rise to the inequality constraints on the control variables $r$ and $c$.

$$r_m \le r(x) \le r_M$$

$$c_m \le c(x) \le c_M \tag{2.2.14}$$

2.3     Methods of Solution

A. Hamilton-Jacobi Equations via Dynamic Programming:

Dynamic Programming is presented here to give some idea about the complexity involved in the numerical solutions of the two point boundary value problem one may face in using techniques that lead to a set of necessary conditions for optimality.

Let us define a new independent variable

$$z = L - x .\tag{2.3.1}$$

The state equation which was the same as (2.12) now becomes

$$\frac{d}{dz} y(z) = - A(u(z))y(z)\tag{2.3.2}$$

and the end point conditions specified in (2.11) now become initial conditions

$$y(z=0) = y_L = y^0 .\tag{2.3.3}$$

With the criterion function as $\phi = \phi(y(z=L))$ we have a Mayer formulation of the variational problem. Bellman and Dreyfus [19] have used a heuristic approach that is very revealing. The optimal payoff function as designated by J, is an implicit function of the initial state $y^0 = y(z_0)$ and the length of the process

$$s = L - z_0 .\tag{2.3.4}$$

The optimal payoff J is defined by

$$J = J(y^0, s) = \max_u [\phi(y(L))] . \qquad (2.3.5)$$

The optimal vector $u^*$ (z) and the optimal state vector $y^*$ (z) all depend on $y^0$ and s. Consider a distance z along the optimal trajectory as shown in Fig. 2.5.
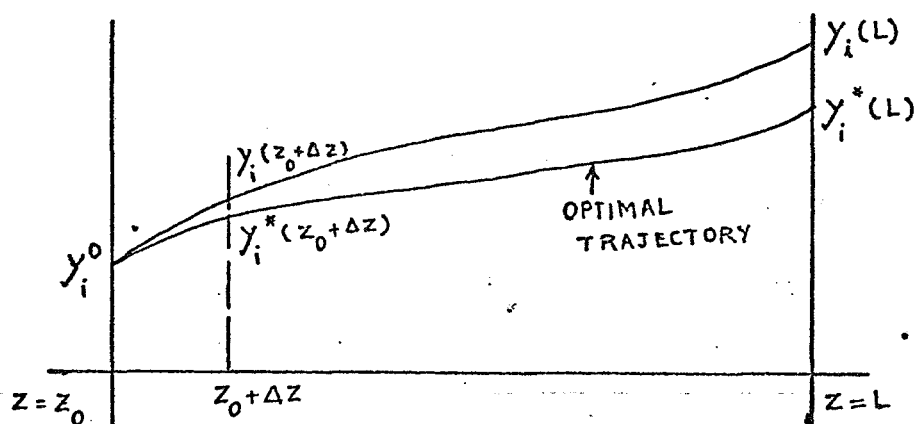


FIGURE 2.5

THE OPTIMAL AND NON-OPTIMAL TRAJECTORIES

The trajectory for nonoptimal u is also shown. Along the optimum path,

$$J(y^0, s) = J(y^*(z_0 + \Delta z), s - \Delta z) , \qquad (2.3.6)$$

since we will end up at $y^*$ (L) along the optimum path, no matter where we start from.

If we take an arbitrary u from $z_0$ to $z_0 + \Delta z$ and an optimal $u = u^*$ from $z_0 + \Delta z$ to L then the payoff function will be $J(y(z_0 + \Delta z), s - \Delta z)$.

Thus,

$$J(y^o, s) = \max_u [J(y(z_0 + \Delta z), s - \Delta z] \quad . \tag{2.3.7}$$

Expanding the right side in a Taylor series and neglecting second and higher order terms,

$$J(y^o, s) = \max_u [J(y^o, s) - \frac{\partial J}{\partial s} \Delta z + \sum_i \frac{\partial J}{\partial y_i} \Delta y_i]$$

$$= \max_u [J(y^o, s) - \frac{\partial J}{\partial s} \Delta z + \sum_i \frac{\partial J}{\partial y_i} \frac{d}{dz} y_i \Big|_{(y^o, z_0)} \Delta z] \quad , \tag{2.3.8}$$

since

$$\Delta Y_j = \frac{d}{dz} y_j \Delta z \quad . \tag{2.3.9}$$

The term on the right side which depends on u is $\frac{d}{dz} y_i(z_0)$. Hence we take all the other terms outside of the bracket and divide by $\Delta z$. Taking the limit as $\Delta z \to 0$ we obtain

$$\frac{\partial J}{\partial s} = \max_{u(z_0)} [\sum_j \frac{\partial J}{\partial y_j} \frac{d}{dz} y_j \Big|_{(y^o, z_0)}] \tag{2.3.10}$$

In (2.3.6) the maximization was to be carried out over the interval $z_0$ to $z_0 + \Delta z$. With $\Delta z \to 0$ the control vector becomes just $u(z_0)$.

The partial differential equation (2.3.10) is true for any z along the trajectory and the corresponding duration s. Thus,

$$\frac{\partial J}{\partial s} = \max_{u(z)} [\sum_i \frac{\partial J}{\partial y_j} \frac{d}{dz} y_j \Big| (y, z)] \quad . \tag{2.3.11}$$

This is a Hamilton-Jacobi equation.

For our case y is a 4-vector and u is a 2-vector, $u^t = [r(z), c(z)]$. We can incorporate the constraint (2.2.12) into the criterion function by means of a Lagrange multiplier $\mu$ and write a new payoff function,

$$\phi = y_1(z=L) + \mu\, y_2(z=L) \quad . \qquad (2.3.12)$$

In order to solve (2.3.7) numerically, we have to discretize it in z. As a first step we have to obtain solution of $u(z_0)$ and then for $u(z_0 + \Delta z)$, assuming $u(z_0)$ to be constant from $z_0$ to $z_0 + \Delta z$, and so on. This becomes a problem of grid formation [20] in five dimensional space $[y_1, y_2, y_3, y_4, s]$. The undetermined Lagrange multiplier $\mu$ is an unknown quantity that has to be determined by trial and error.

No attempt was made to obtain the numerical solutions using this approach, since the curse of dimensionality is indefeatable.

B.  Pontryagin's Maximum Principle:

The Maximum Principle will yield a set of necessary conditions that the optimal control u* has to satisfy if such control exists, and if it optimizes the criterion function.

Given the system equations (restatement of (2.3.2) and (2.3.3)),

$$\frac{d}{dz} y(z) = -A(u(z))y(z) \quad , \qquad (2.3.13)$$

with boundary conditions

$$y(z=0) = y^0 = [a,0,0,0] \quad , \qquad (2.3.14)$$

the Maximum Principle states that in order that the trajectory y(z) be optimal in the sense that the criterion function is maximized it is necessary that one can find functions $\lambda(z)$, defined as the adjoint variables, satisfying the following properties,

(i) the $\lambda_j$ satisfy the differential equations,

$$\frac{d}{dz}\lambda_j + \sum_{i=1}^{4}[\frac{d}{dy_j}f_i(y^*(z),u^*(z))]\lambda_i(z) = 0 \quad (2.3.15)$$

(ii) letting

$$E(u) = \sum_{i=1}^{4} f_i(y(z),u)\lambda_i(z)$$

where u is arbitrary, nonoptimal 2-vector, then

$$E(u^*(z)) \geq E(u(z)) \quad \text{for all z and admissible u.}$$

$$(2.3.16)$$

At this stage we may introduce the Hamiltonian,

$$H(y,u,\lambda) = \sum_{j=1}^{4} \lambda_j f_j(y,u) \quad (2.3.17)$$

Thus, in terms of the Hamiltonian the Maximum Principle states that, given

$$\frac{d}{dz}y = H_\lambda^t$$

and $\quad \frac{d}{dz}\lambda = -H_y^t$ , then $\quad (2.3.18)$

in order that y (z) maximizes the criterion function it is necessary that

$$H(y(z), u(z), \lambda(z)) \geq H(y(z), \bar{u}, \lambda(z)) \qquad (2.3.19)$$

where $\bar{u}$ is any constant admissible control vector. Also the adjoint variables have to satisfy the transversality condition at $z=L$

$$\lambda_j(L) = \frac{\partial \phi}{\partial y_j} - \mu \frac{\partial \Omega}{\partial y_j} \quad , \qquad (2.3.20)$$

where $\phi$ is the criterion function, $\Omega$ is the boundary constraint and $\mu$ is the undetermined Lagrange Multiplier.

Coming back to our system of equations,

$$\frac{d}{dz} y = - Ay$$

$$\frac{d}{dz} \lambda = A^t \lambda$$

$$H = - \lambda^t Ay$$

$$y^t(0) = [a,0,0,0]$$

$$\lambda^t(L) = [1,-\mu,0,0] \quad . \qquad (2.3.21)$$

With the inequality constraint (2.2.14) on the control, (2.3.19) gets transformed into

$$\frac{\partial H}{\partial u^*_i} > 0 \qquad \text{if } u^*_i = u_{imax} \quad ,$$

$$\frac{\partial H}{\partial u^*_i} = 0 \qquad \text{if } u_{imax} > u^*_i > u_{imin} \quad ,$$

$$\frac{\partial H}{\partial u^*_i} < 0 \qquad \text{if } u^*_i = u_{imin} \quad , \qquad (2.3.22)$$

and the constraining equation

$$\Omega[y(L)] = 0 . \tag{2.3.23}$$

Equations (2.3.21) to (2.3.23) comprise a self-sufficient set of equations, which, when solved will yield the optimal control $u^*(z)$. This is a two point boundary value problem.

Since $A(u(z))$ is linear in u, H turns out to be linear in u.

$$H_r = (\lambda_3 y_3 + \lambda_2 y_4) ,$$

$$H_c = \omega(\lambda_3 y_2 - \lambda_4 y_1) . \tag{2.3.24}$$

Thus (2.3.19) suggests a bang bang control. In other words, we are tempted to believe that $u_i$ will always be at either boundary and will switch whenever $H_{u_i}$ changes sign.

Assuming such a bang bang form of the control, a combination of iterations and scanning (for $\mu$) was used to find a solution. The iterations did not converge. The solutions obtained by the gradient technique which is described later, indicate that the assumption regarding the form of the control was erroneous. It does not turn out to be bang bang control. One way of circumventing this is by adding to the criterion function a penalizing functional [4] that is nonlinear in control.

As Johnson and Gibson [13] have pointed out, it is characteristic of the solutions to linear optimization problems that the switching function $H_{u_i}$ sometimes become identically zero over

some finite interval of 'z'. Since, during this interval, H does not depend upon u explicitly, the usual procedure of selecting u* so as to maximize H breaks down. These linear optimization problems where $H_{u_i}$ becomes identically zero over finite interval have been referred to as "singular". It has been shown that the optimal control may actually consist of intervals of variable control effort (called "singular switching curves") combined with intervals of limiting control.

Thus, there seems to be a distinct possibility of the optimal control being a limiting control with singular curves rather than a bang bang control with switching points.

## C. Gradient Technique [14]: First Order

The approaches described so far are based on obtaining a set of necessary conditions for optimality and then trying to get solutions to this set of equations. The alternate approach, as already mentioned in the previous chapter, is the "Gradient Technique". Consider a small perturbation $\delta u(x)$ in the control variable with reference to the set of equations (1.2.10).

$$\frac{d}{dx} y_i = f_i(y(x), u(x)) \ .$$

The resulting first order variational equation is given by

$$\frac{d}{dx} (\delta y_i) = \sum_j \frac{\partial f_i}{\partial y_j} \delta y_j + \sum_k \frac{\partial f_i}{\partial u_k} \delta u_k \ . \qquad (2.3.25)$$

Define a set of adjoint variables by the differential equations,

$$\frac{d}{dx}(\lambda_i) = -\sum_j \frac{\partial f_j}{\partial y_i} \delta \lambda_j \quad . \tag{2.3.26}$$

Multiplying (2.3.25) by $\lambda_i$, (2.3.26) by $\delta y_i$, adding them together and performing summation over index i,

$$\frac{d}{dx}\left[\sum_i \lambda_i \delta y_i\right] = \sum_i \sum_k \lambda_i \frac{\partial f_i}{\partial u_k} \delta u_k \quad . \tag{2.3.27}$$

Define

$$H = \langle \lambda, f \rangle = \sum_i \lambda_i f_i \quad ,$$

$$H_{u_k} = \sum_i \lambda_i \frac{\partial f_i}{\partial u_k} \quad .$$

Then, integrating (2.3.27) from x=0 to x=L,

$$\left[\sum_i \lambda_i \delta y_i\right]_0^L = \int_0^L H_u \delta u \, dx \quad . \tag{2.3.28}$$

Since y(L) is completely specified by (2.2.11)

$$\delta y(L) = 0 \quad .$$

Here we can define $\lambda^\phi$ and $\lambda^\Omega$ as the adjoint system variables satisfying (2.3.26) subject to boundary conditions

$$\lambda_i^\phi(0) = -\frac{\partial \phi}{\partial[y_i(0)]} \tag{2.3.29}$$

and

$$\lambda_i^\Omega(0) = -\frac{\partial \Omega}{\partial[y_i(0)]} \tag{2.3.30}$$

respectively. When $\phi$ is given by (2.2.13), equation (2.3.29) becomes

$$\lambda^{\phi}(0) = [-1,0,0,0]^t \quad .$$

Similarly, when $\Omega$ is given by (2.2.12), equation (2.3.30) becomes

$$\lambda^{\Omega}(0) = [0,-1,0,0]^t \quad .$$

Now we can define

$$H^{\phi} = <\lambda^{\phi},f>$$

and

$$H^{\Omega} = <\lambda^{\Omega},f> \quad .$$

Substituting (2.3.29) into (2.3.28), we obtain

$$d\phi = \int_u H^{\Omega}\delta u dx \qquad\qquad (2.3.31)$$

Similarly, substituting (2.3.30) into (2.3.28) we obtain

$$d\Omega = \int_u H^{\Omega}\delta u dx \qquad\qquad (2.3.32)$$

Equations (2.3.31) and (2.3.32) give the functional relationship between variation in control, and change in criterion functional and constraint in response to it.

The initial arbitrary non-optimal choice of u or the subsequent estimates of u during iteration process may not exactly satisfy the constraint $\Omega=0$. Therefore, at every stage there are two variations required.

(i) Change $d\phi$ in order to improve the criterion function.

(ii)  Change $d\Omega = -\Omega$ in order to satisfy the constraining equation, thus making $\Omega + d\Omega = 0$.

An accessory minimization problem is formed by stipulating an arbitrary criterion functional

$$\phi_s = \frac{1}{2} \int \delta u^t(x) W(x) \delta u(x) dx$$

which has to be minimized while satisfying (2.3.31) and (2.3.32). [See equation (1.3.24) and the following paragraph.] In order to obtain the desired variation in the criterion functional $\phi$ and the constraint $\Omega$ while minimizing the above criterion function, we form a composite criterion functional for the accessory minimization problem. We seek a $\delta u$ that minimizes the composite criterion functional

$$\psi = \frac{1}{2} \int \delta u^t W \delta u \, dx + \nu^\phi [d\phi - \int H_u^\phi \delta u \, dx] + \nu^\Omega [d\Omega - \int \delta u \, dx]. \quad (2.3.33)$$

where $\nu^\phi$ and $\nu^\Omega$ are undetermined Lagrange multipliers to be chosen so as to satisfy (2.3.31) and (2.3.32).
Euler Lagrange equations give

$$\delta u = W^{-1} [\nu^\phi H_u^{\phi t} + \nu^\Omega H_u^{\Omega t}] \quad . \quad (2.3.34)$$

Defining

$$\delta u^\phi = W^{-1} H_u^{\phi t} \quad (2.3.35)$$

$$\delta u^\Omega = W^{-1} H_u^{\Omega t} \quad (2.3.36)$$

with

$$H_u^t = \begin{array}{c} -(\lambda_1 y_3 + \lambda_2 y_4) \\ \omega(\lambda_4 y_1 - \lambda_3 y_2) \end{array}$$

therefore $\delta u = \nu^\phi \delta u^\phi + \nu^\Omega \delta u^\Omega$ (2.3.37)

Substituing for $\delta u$ in (2.3.31) and (2.3.32) we get finally

$$d\phi = \nu^\phi \int H_u^\phi \delta u^\phi dx + \nu^\Omega \int H_u^\phi \delta u^\Omega dx \qquad (2.3.38)$$

and

$$d\Omega = \nu^\phi \int H_u^\Omega \delta u^\phi dx + \nu^\Omega \int H_u^\Omega \delta u^\Omega dx \qquad (2.3.39)$$

The iteration algorithm that is suggested by the above equations is fairly straightforward and proceeds as follows.

1) Assume a nominal control $u$. Solve the system equations (2.2.9) with boundary conditions (2.2.11).

2) Solve the adjoint equations (2.3.26): (i) with boundary condition (2.3.29) to obtain $\lambda^\phi(x)$ and (ii) with boundary condition (2.3.30) to obtain $\lambda^\Omega(x)$.

3) Evaluate $\delta u^\phi$ and $\delta u^\Omega$ from (2.3.35) and 2.3.36), with $W^{-1}$ given.[5]

4) Solve (2.3.38) and (2.3.39)[6] for $\nu^\phi$ and $\nu^\Omega$.

5) Evaluate $\delta u$ from (2.3.37).

---

[5] $W^{-1}$ is chosen based on knowledge of the system, and could be made equal to the unity matrix.

[6] $d\phi$ and $d\Omega$ must be chosen beforehand. $d\phi$ is chosen for convergence and $d\Omega$ to satisfy the constraint equation (2.2.12).

6) Add $\delta u$ to u and obtain a revised estimate for the control as $u + \delta u$. The inequality constraints on u are taken into account by truncating $(u + \delta u)$ at $u_M$ or $u_m$ wherever it attempts to exceed the bounds. The validity of truncation in connection with convergence can be proven for the $\phi$ and $\Omega$ corrections separately (see Appendix B). A more detailed discussion is included in Section 2.4.

7) Now branch back to the start of the loop for the next iteration cycle.

D. Gradient Technique: Second Order Improvement

For an open region for u, the convergence of the first order gradient technique in the neighborhood of the optimal solution has always been poor. The reason for poor convergence is that as a necessary condition for the optimality the influence function $H_u$ tends to zero near the optimal trajectory. Second order terms take a dominating role in this neighborhood. Bullock [21] has developed a method that takes into account the second order variations. Bullock considers the second order terms in the variational equation relating a variation in the control to the variation in the criterion functional and then solves the two point boundary value problem generated as a solution to the accessory optimization problem.

It is felt by this investigator that there is enough information available in the solutions to the state and adjoint

equations that can be made use of in obtaining second order

estimates of $\delta u$.

The first part of the development handles a general case

where the form for $f(y,u)$ in the system equations

$$\frac{d}{dx} y = f(y,u) \qquad (2.3.40)$$

is not specified. In part (ii) a specific form of $f$ is

assumed so that the results can be directly applicable to the

system representing a transmission line.

(i) General Case

The system equations are given by (2.3.40) with the boundary

conditions

$$y(x=L) = y_L \quad . \qquad (2.3.40a)$$

The criterion function is

$$\phi = \phi(y(x=0)) \quad . \qquad (2.3.40b)$$

and the constraint is

$$\Omega = \Omega(y(x=0)) = 0 \quad . \qquad (2.3.40c)$$

In general, the boundary conditions may be specified partly at

x=0 and partly at x=L. In such a case the criterion function $\phi$

and the constraint $\Omega$ also can be the functions of the terminal

values of y at both ends, x=0 and x=L. The particular case

described by (2.3.40) to (2.3.40c) is chosen because it is similar

to the trasmission line problem we will be handling later.
However, the results can be extended, without any difficulty to
the case where the boundary conditions are mixed. In order to
derive a functional relationship between a variation in control
u and a corresponding variation in the criterion functional $\phi$,
let us consider a small perturbation in the control vector u.
The variational equation may be obtained as,

$$\frac{d}{dx} (\delta y) = f_y \delta y + f_u \delta u + \frac{1}{2} \delta y^t f_{yy} \delta y + \delta y^t f_{yu} \delta u + \frac{1}{2} \delta u^t f_{uu} \delta u$$
$$+ f_h (y,u,\delta y,\delta u) , \qquad (2.3.41)$$

where $f_h$ represents all the higher order terms in the expansion.
As before define a set of adjoint variables such that,

$$\frac{d}{dx} \lambda = - f_y^t \lambda . \qquad (2.3.42)$$

Multiplying (2.3.41) by $\lambda^t$ and (2.3.42) by $(\delta y^t)$ and adding
them we obtain

$$\frac{d}{dx} [\lambda^t \delta y] = \lambda^t f_u \delta u + \frac{1}{2} \lambda^t \delta y^t f_{yy} \delta y + \lambda^t \delta y^t f_{yu} \delta u + \frac{1}{2} \lambda^t \delta u^t f_{uu} \delta u$$
$$+ \lambda^t f_h (y,u,\delta y,\delta u) .$$

The Hamiltonian is defined as

$$H = \lambda^t f ,$$

also

$$H_h = \lambda^t f_h .$$

Now the equation is integrated from $x=0$ to $x=L$, to obtain

$$[\lambda^t \delta y]_0^L = \int_0^L [H_u \delta u + \frac{1}{2} \delta y^t H_{yy} \delta y + \delta y^t H_{yu} \delta u + \frac{1}{2} \delta u^t H_{uu} \delta u$$

$$+ H_h (y,u,\delta y,\delta u)] dx \ . \qquad (2.3.43)$$

If we set the boundary conditions on $\lambda$ such that

$$\lambda^\Omega_{(x=0)} = - \Omega^t_y \Big|_{x=0} \qquad (2.3.44)$$

and neglect the terms higher than second order, the equation (2.3.43) can be rewritten as

$$d\phi = \int_0^L H_u^\phi \delta u \, dx + \int_0^L \delta y^t H_{yu}^\phi \delta u \, dx + \frac{1}{2} \int_0^L (\delta y^t H_{yy} \delta y + \delta u^t H_{uu} \delta u) dx \ . \qquad (2.3.45)$$

Similarly, if we define another set of boundary conditions corresponding to constraint $\Omega$,

$$\lambda^\Omega_{(x=0)} = \Omega^t_y \Big|_{x=0} \ , \qquad (2.3.46)$$

we have

$$d\Omega = \int_0^L H_u^\Omega \delta u \, dx + \int_0^L \delta y^t H_{yu}^\Omega \delta u \, dx + \frac{1}{2} \int_0^L (\delta y^t H_{yy}^\Omega \delta y$$

$$+ \delta u^t H_{uu}^\Omega \delta u) dx \ . \qquad (2.3.47)$$

The equations (2.3.45) and (2.3.47) are the second order estimates of the variations in criterion function $\phi$ and constraint function $\Omega$, resulting from a variation $\delta u$ in the control variable, as a functional of the nominal trajectory corresponding to a nominal $u$

and the variations $\delta y$. The nominal trajectory will yield certain

value of $\phi$ and most probably a nonzero value of $\Omega$ . One can pick

(i) a desired value of $d\phi$ so as to improve $\phi$, and

(ii) $d\Omega = -\Omega$ , so as to make the new trajectory fulfill

the constraining equation, $\Omega = 0$,

and hope to find a set of functions $\delta u$ with the help of the

functional relationship given by (2.3.45) and (2.3.47). The

second order variational equation for $\delta y$ is given by (2.3.41)

without the term $f_h$.

We have a problem of determining the best $\delta u$, in some sense,

so that the trajectory $\delta y$ as governed by

$$\frac{d}{dx}(\delta y) = f_y \delta y + f_u \delta u + \frac{1}{2} \delta y^t f_{yy} \delta y + \delta y^t f_{yu} \delta u + \frac{1}{2} \delta u^t f_{uu} \delta u$$

$$= g(\delta u, \delta y) \qquad (2.3.48)$$

with the boundary conditions $\delta y_{(x=L)} = 0$

satisfies the functional equations (2.3.45) and (2.3.47).

The problem, as has been referred to previously in part C

of Section 2.3, is an accessory optimization problem. The

criterion function for the accessory problem is chosen as

$$\phi_s = \frac{1}{2} \int \delta u^t(x) W(x) \, \delta u \, (x) dx$$

For the accessory minimization problem, $\delta y$ is the state variable

and $\delta u$ is the control variable. Equation (2.3.46) is the system

equation and (2.3.45) and (2.3.47) are the constraining equations.

Incorporating the system equations and the constraining equations into the criterion function (as in Part C), we get the modified criterion function

$$\phi_m = \frac{1}{2} \int \delta u^t W \delta u \, dx + v^\phi [d\phi - \int (H_u^\phi + \delta y^t H_{yu}^\phi) \delta u \, dx$$

$$- \frac{1}{2} \int (\delta y^t H_{yy} \delta y + \delta u^t H_{uu} \delta u) dx]$$

$$+ v^\Omega [d\Omega - \int (H_u^\Omega + \delta y^t H_{yu}^\Omega) \delta u \, dx + \frac{1}{2} \int (\delta y^t H_{yy} \delta y$$

$$+ \delta u^t H_{uu} \delta u) dx] + \int \delta \lambda^t [\frac{d}{dx} (\delta y) - g(\delta y, \delta u)] dx \qquad (2.3.49)$$

where $v^\phi$, $v^\Omega$, and $\delta\lambda(x)$ are the Lagrange multipliers. The choice of $\delta\lambda$ as the Lagrange multipliers for the system equations is by no means an accidental choice. As will be seen from equation (2.3.51) which defines the differential equations for $\delta\lambda$, the variable $\delta\lambda$ is adjoint to the system variables $\delta y$ in the same sense as the adjoint variables $\lambda$ are adjoint to the system variables y [see (1.3.1) and (1.3.3)]. We will use the indirect method of the calculus of variations to obtain the set of necessary conditions for optimality. An iterative technique is developed for solving these equations. The Euler Lagrange equation for $\delta u$ is,

$$\delta u = (W - v^\phi H_{uu}^\phi - v^\Omega H_{uu}^\Omega - H_{\lambda uu} \delta \lambda)^{-1} [v^\phi (H_u^\phi + H_{yu}^\phi \delta y)^t$$

$$+ v^\Omega (H_u^\Omega + H_{yu}^\Omega \delta y)^t + (H_{\lambda u} \delta \lambda + \delta y H_{\lambda uy} \delta \lambda)^t] \quad . \qquad (2.3.50)$$

The Euler Lagrange equations for system variables y yield the adjoint system equation

$$\frac{d}{dx} \delta\lambda = -(f_y^t + f_{yu}^t \delta u + f_{yy}^t \delta y)\delta\lambda$$

$$- \nu^\phi(H_{yu}^\phi \delta u + H_{yy}^\phi \delta y) - \nu^\Omega(H_{yu}^\Omega \delta u + H_{yy}^\Omega \delta y) \ . \qquad (2.3.51)$$

The transversality conditions yield the boundary conditions

$$\delta\lambda(x=0) = 0 \ .$$

The complete set of equations required to be solved for obtaining $\delta u$, is now given by (2.3.48), (2.3.51), and (2.3.50) with (2.3.45) and (2.3.47). These are coupled differential equations with mixed boundary conditions. The situation appears to be hopelessly complicated.

However, it is observed that the y and $\lambda$ trajectories for u and u+$\delta$u are related to $\delta$y and $\delta\lambda$ in a simple manner. This relationship can be made use of in developing an iterative technique to obtain the approximate solutions to the set of necessary conditions for the accessory minimization problem.

Let us consider the $\lambda^\phi$ system as defined by (2.3.26)

$$\frac{d}{dx} \lambda^\phi = -f_y^t \lambda^\phi$$

The variation in this system due to $\delta$u is given by

$$\frac{d}{dx}(\delta\lambda^\phi) = -(f_y^t + f_{yu}^t \delta u + f_{yy}^t \delta y)\delta\lambda^\phi$$

$$- f_{yy}^t \delta y \lambda^\phi - f_{yu}^t \delta u \lambda^\phi$$

$$= -(f_y^t + f_{yu}^t \delta u + f_{yy}^t \delta y)\delta\lambda^\phi$$

$$- (H_{yu}^\phi \delta u + H_{yy}^\phi \delta y) \qquad (2.3.52)$$

The variation $\delta\lambda^\phi$ is not yet related to $\delta\lambda$. Thus $\delta y$ is not an implicit function of $\delta\lambda^\phi$. The other coefficients of $\delta\lambda^\phi$ in (2.3.52) are functions of the trajectories $y$ and $\lambda$ corresponding to a nominal control $u$. Now, if the estimate of $\delta u$ is known (2.3.52) becomes a linear non-homogeneous differential equation for $\delta\lambda^\phi$. The factor $(H_{yu}^\phi \delta u + H_{yy}^\phi \delta y)$ is a forcing function in (2.3.52). Since $\lambda^\phi$ is completely specified at $x=0$, the boundary conditions are

$$\delta\lambda^\phi(x=0) = 0$$

Similarly, we may obtain

$$\frac{d}{dx}(\delta\lambda^\Omega) = -(f_y^t + f_{yu}^t \delta u + f_{yy}^t \delta y)\delta\lambda^\Omega - (H_{yu}^\Omega \delta u + H_{yy}^\Omega \delta y)$$

and

$$\delta\lambda^\Omega(x=0) = 0 \ . \qquad (2.3.53)$$

Equations (2.3.51), (2.3.52), and (2.3.53) are the same differential equations with identical boundary conditions differing only in the forcing functions. It may be noted that the solutions to

a set of linear differential equations with zero boundary

conditions are linearly dependent upon the forcing functions.

Making use of this property we can write

$$\delta\lambda = \nu^\phi\delta\lambda^\delta + \nu^\Omega\delta\lambda^\Omega \quad , \qquad (2.3.54)$$

where $\delta\lambda^\phi$ and $\delta\lambda^\Omega$ are variations in $\lambda^\phi$ and $\lambda^\Omega$ due to u change

in control variable. As has been stated before, we hope to

extract the information already available without having to

simulate and solve any systems other than those required for

the first order estimate of $\delta u$. The variables $\delta\lambda^\phi$ and $\delta\lambda^\Omega$ are

the variations in $\lambda^\phi$ and $\lambda^\Omega$ systems corresponding to a variation

$\delta u$. The solutions to $\lambda^\phi$ and $\lambda^\Omega$ systems with u+$\delta$u as the control

will yield these variations. The variation $\delta\lambda$ can be estimated

with the help of (2.3.52). Thus we have eliminated the necessity

for solving the set of equations (2.3.51) in a direct way. This

property is of paramount importance in the Hybrid Computations.

As will be shown in Section 2.4 the systems y, $\lambda^\phi$, and $\lambda^\Omega$ can

each be represented by the same analog computer patching with

the help of the proper transformations and change of variable.

Any additional system representation causes considerable compli-

cations. With (2.3.54) as a new estimate of $\delta\lambda$, (2.3.50) can be

rewritten as

$$\delta u = [W - \nu^\phi(H^\phi_{uu} + H_{\lambda uu}\delta\lambda^\phi) - \nu^\Omega(H^\Omega_{uu} + H_{\lambda uu}\delta\lambda^\Omega)]^{-1}$$

$$[\nu^\phi(H^\phi_u + H^\phi_{yu}\delta y + H_{\lambda u}\delta\lambda^\phi + \delta y H_{\lambda uy}\delta\lambda^\phi)$$

$$+ \nu^\Omega(H^\Omega_u + H^\Omega_{yu}\delta y + H_{\lambda u}\delta\lambda^\Omega + \delta y H_{\lambda uy}\delta\lambda^\Omega)] \quad . \qquad (2.3.55)$$

The problem cannot be simplified any further with f(y,u) as a general vector function. Let us restrict ourselves to a more specific case which covers the transmission line problem.

(ii)  Second Order System Equations

Let us restrict the form of f(y,u) so that the system equations are of second order in y and u, i.e.

$$\frac{\partial f^3}{\partial y_i \partial y_j \partial u_k} \equiv \frac{\partial f^3}{\partial u_i \partial u_j \partial y_k} \equiv 0 \quad,$$

where i, j, and k take integral values. Thus the terms of the order higher than two in the variational equations developed so far will vanish identically. The second order variational equations become an exact description of the second order system rather than being the approximate ones.

Now we can outline an iteration procedure to obtain successive approximations for δu as a solution of the accessory minimization problem. Assuming the knowledge of δu, say from the first order estimation, the variational functions δy, δλ$^\phi$ and δλ$^\Omega$ can be obtained as

$$\delta y = y\big|_{(u+\delta u)} - y\big|_{(u)} \quad, \tag{2.3.56}$$

$$\delta \lambda^\phi = \lambda^\phi\big|_{(u+\delta u)} - \lambda^\phi\big|_{(u)} \quad, \tag{2.3.57}$$

$$\delta \lambda^\Omega = \lambda^\Omega\big|_{(u+\delta u)} - \lambda^\Omega\big|_{(u)} \quad. \tag{2.3.58}$$

In order to get the new estimate of $\delta u$ we can use the relationships

$$d\phi = \int H_u^\phi \Big|_{(y+\delta y,\, u+\delta u)} \delta u\, dx \, , \qquad (2.3.59)$$

$$= \int H_u^\phi \Big|_{(y+\delta y,\, u)} \delta u\, dx + \int \delta u^t H_{uu}^t \Big|_{(y+\delta y,\, u)} \delta u\, dx \qquad (2.3.60)$$

Similarly,

$$d\Omega = \int H_u^\Omega \Big|_{(y+\delta y,\, u)} \delta u\, dx + \int \delta u^t H_{uu}^t \Big|_{(y+\delta y,\, u)} \delta u\, dx \qquad (2.3.61)$$

and

$$\delta u = [W - \nu^\phi H_{uu}^\phi \Big|_{(\lambda^\phi + \delta\lambda^\phi)} - \nu^\Omega H_{uu}^\Omega \Big|_{(\lambda^\Omega + \delta\lambda^\Omega)}]$$

$$(\nu^\phi \delta u^\phi + \nu^\Omega \delta u^\Omega) \qquad (2.3.62)$$

where

$$\delta u^\phi = (H_u^\phi \Big|_{(y+\delta y,\, \lambda^\phi + \delta\lambda^\phi)})^t \, , \qquad (2.3.63)$$

$$\delta u^\Omega = (H_u^\Omega \Big|_{(y+\delta y,\, \lambda^\Omega + \delta\lambda^\Omega)})^t \, . \qquad (2.3.64)$$

The equations (2.3.60), (2.3.61), and (2.3.62) are the restatements of (2.3.45), (2.3.47), and (2.3.55) respectively. The desired improvements $d\phi$ and $d\Omega$ are specified. Hence the substitution of (2.3.62) in (2.3.60) and (2.3.61) yield a pair of simultaneous equations for $\nu^\phi$ and $\nu^\Omega$. The values of $\nu^\phi$ and $\nu^\Omega$ obtained from these simultaneous equations are substituted in (2.3.62) to obtain a new estimate of $\delta u$. With this $\delta u$ one returns to equations (2.3.56), (2.3.57), and (2.3.58) and repeats the

cycle. This is a secondary iteration loop for obtaining a second order estimate of $\delta u$ at a given nominal $\delta u$. After $v^\phi$, $v^\Omega$, and $\delta u$ "converge" to their correct values (see Part B of Section 2.4) $u+\delta u$ is formed. This is a new estimate of $u$ for the new iteration cycle of the main iteration loop.

(iii) Phase-Shift Oscillator Equations

Now that we have all the tools for choosing $\delta u$, let us apply the method to phase-shift oscillator. In the case of a phase-shift oscillator the situation is simpler and much more manageable. From equation (2.2.9)

$$\frac{\partial^2 f}{\partial y_i \partial y_j} \equiv \frac{\partial^2 f}{\partial u_i \partial u_j} \equiv 0 \quad .$$

Hence all the terms in $H_{yy}$ and $H_{uu}$ vanish identically. Letting

$$\delta u^\phi = W^{-1}(H_u^\phi |_{(y+\delta y, \lambda^\phi + \delta \lambda^\phi)})^t \quad , \tag{2.3.66}$$

$$\delta u^\Omega = W^{-1}(H_u^\Omega |_{(y+\delta y, \lambda^\Omega + \delta \lambda^\Omega)})^t \quad , \tag{2.3.67}$$

equation (2.3.62) assume the form,

$$\delta u = v^\phi \delta u^\phi + v^\Omega \delta u^\Omega \quad . \tag{2.3.68}$$

Here we will make an approximation and neglect the second order terms in equation (2.3.48). Therefore,

$$\frac{d}{dx} \delta y = f_y \delta y + f_{yu} \delta u \quad . \tag{2.3.69}$$

Let us define the $y_\phi$ system as

$$\frac{d}{dx} \delta y_\phi = f_y \delta y_\phi + f_{yu} \delta u^\phi \;,$$

(2.3.70)

$$\delta y_{\phi(x=0)} = 0 \;,$$

and the $y_\Omega$ system as

$$\frac{d}{dx} \delta y_\Omega = f_y \delta y_\Omega + f_{yu} \delta u^\Omega \;,$$

(2.3.71)

$$\delta y_{\Omega(x=0)} = 0 \;.$$

Then noting that $\delta y_{(x=0)} = 0$ and the systems (2.3.69), (2.3.70), and (2.3.71) are linear, we can write,

$$\delta y = v^\phi \delta y_\phi + v^\Omega \delta y_\Omega \;,$$

(2.3.72)

where

$$\delta y_\phi = y\big|_{(u+\delta u^\phi)} - y\big|_{(u)} \;,$$

(2.3.73)

$$\delta y_\Omega = y\big|_{(u+\delta u^\Omega)} - y\big|_{(u)} \;.$$

(2.3.74)

Substituting for $\delta y$ from (2.3.72) and for $\delta u$ from (2.3.68) the equation (2.3.60) takes a form

$$d\phi = v^\phi \int H_u^\phi \delta u^\phi dx + v^\Omega \int H_u^\phi \delta u^\Omega dx + v^{\phi^2} \int \delta y_\phi^t H_{yu}^\phi \delta u^\phi dx$$

$$+ v^\phi v^\Omega \Big[ \int \delta y_\phi^t H_{yu}^\phi \delta u^\Omega dx + \int \delta y_\Omega^t H_{yu}^\phi \delta u^\phi dx \Big]$$

$$+ v^{\Omega^2} \int \delta y_\Omega^t H_{yu}^\phi \delta u^\Omega dx \;.$$

(2.3.75)

Similarly, from (2.3.61)

$$d\Omega = \nu^\phi \int H_u^\Omega \delta u^\phi dx + \nu^\Omega \int H_u^\Omega \delta u^\Omega dx + {\nu^\phi}^2 \int \delta y_\phi^{t} H_{yu}^\Omega \delta u^\phi dx$$

$$+ \nu^\phi \nu^\Omega [\int \delta y_\phi^{t} H_{yu}^\Omega \delta u^\Omega dx + \int \delta y_\Omega^{t} H_{yu}^\Omega \delta u^\phi dx]$$

$$+ {\nu^\Omega}^2 \int \delta y_\Omega^{t} H_{yu}^\Omega \delta u^\Omega dx \quad . \tag{2.3.76}$$

If we know the nominal trajectories and first estimates of $\delta u^\phi$, $\delta u^\Omega$, $\delta y_\phi$, and $\delta y_\Omega$, the above two equations can be solved for $\nu^\phi$ and $\nu^\Omega$. Equation (2.3.55) can be approximated by

$$\delta u^\phi = H_u^{\phi t} + H_{yu}^\phi \delta y + H_{\lambda u}^\phi \delta \lambda^\phi \quad . \tag{2.3.77}$$

Using the same arguments and approximations as those used for deriving (2.3.73) and (2.3.74) we can derive a similar relationship for the variation in $\lambda^\phi$ system. Thus,

$$\delta \lambda^\phi = \nu^\phi \delta \lambda_\phi^\phi + \nu^\Omega \delta \lambda_\Omega^\phi \quad . \tag{2.3.78}$$

where $\lambda_\phi^\phi$ is the variation in $\lambda^\phi$ system due to the variation $\delta u^\phi$ and $\lambda_\Omega^\phi$ is the variation in $\lambda^\phi$ system due to the variation $\delta u^\Omega$. Substituting (2.3.72) and (2.3.78) in (2.3.77),

$$\delta u^\phi = H_u^{\phi t} + \nu^\phi (H_{yu}^\phi \delta y_\phi + H_{\lambda u} \delta \lambda_\phi^\phi) \tag{2.3.79}$$

$$+ \nu^\Omega (H_{yu}^\phi \delta y_\Omega + H_{\lambda u} \delta \lambda_\Omega^\phi) \quad .$$

Similarly, for the $\lambda^\Omega$ system

$$\delta\lambda^\Omega = \nu^\phi\delta\lambda_\phi^\Omega + \nu^\Omega\delta\lambda_\Omega^\Omega \qquad (2.3.80)$$

and hence

$$\delta u^\Omega = H_u^{\Omega t} + \nu^\phi(H_{yu}^\Omega \delta y_\phi + H_{\lambda u}\delta\lambda_\phi^\Omega)$$

$$+ \nu^\Omega(H_{yu}^\Omega \delta y_\Omega + H_{\lambda u}\delta\lambda_\Omega^\Omega) \ . \qquad (2.3.81)$$

The above two expressions give estimates of $\delta u^\phi$ and $\delta u^\Omega$ in terms of $\nu^\phi$, $\delta y_\phi$, $\delta\lambda^\phi$, $\nu^\Omega$, $\delta y_\Omega$, $\delta\lambda^\Omega$, which in turn depend upon the previous estimate of $\delta u^\phi$ and $\delta u^\Omega$. Thus, we have here a recurrence relationship. Starting with the initial guess of

$$\delta u^\phi = H_u^{\phi t} \ ,$$

$$\delta u^\Omega = H_u^{\Omega t} \ ,$$

one can solve the variational equations for $\delta y$ and $\delta\lambda$, and obtain the corresponding multiplier $\nu^\phi$ and $\nu^\Omega$ from (2.3.75) and (2.3.76). The estimate of $\delta u^\phi$ and $\delta u^\Omega$ may then be updated from (2.3.79) and (2.3.81). After three or four iterations (2.3.68) can be used to estimate $\delta u$ that will effect the desired changes $d\phi$ and $d\Omega$.

In the case of the problem attempted on the Hybrid Computer, reasonable convergence was obtained in three or four iterations. It may be noted that the term convergence has been used here only with reference to $\delta u^\phi$ and $\delta u^\Omega$. The values of $\nu^\phi$ and $\nu^\Omega$ settled

down to what appeared to be their final values for the secondary
iteration loop.

After the secondary loop is completed the $\delta u$ obtained is
added to u. The resultant control u+$\delta u$ is checked for bounds
and truncated at the boundaries if necessary. The main itera-
tion cycle then proceeds with this new u+$\delta u$ as a nominal control.
The second order gradient technique yielded a $\delta u$ that gave the
predicted improvements $d\phi$ and $d\Omega$ as long as the control u
stayed in the open region. Once the control reached the bounds
the estimations started to go wrong. The causes for such a pheno-
menon are discussed in Part A of the next section. The failure
of the second order technique indicated that the convergence
problems are not necessarily associated with the collapse of $H_u$.
Instead, they are associated with the bounded nature of the
control. The following section presents the improved first
order gradient technique that is devised to handle this diffi-
culty. This new technique yielded very satisfactory results.

2.4    Algorithms and Programs

The iterative solutions are obtained on a Hybrid Computer using the improved gradient technique. Appendix A describes the features and certain operations of the Hybrid system. The analog computer is used exclusively for solving the differential equations. The digital computer supplies the continuously varying coefficients. The synchronous operation of the analog and digital computer units yields the solutions to the differential equations. The solutions to the various system equations are stored into the memory of the digital computer and are subsequently operated upon to obtain the desired variation in the control variables. The entire operation is under complete program control of the digital computer.

Analog Patching:

We need to solve three sets of system equations on the analog computer. The system proper is described by (from (2.3.2)),

$$\frac{d}{dz} y_1(z) = r(z)y_3(z) + \omega \ell(z)y_4(z) \quad ,$$

$$\frac{d}{dz} y_2(z) = r(z)y_4(z) - \omega \ell(z)y_3(z) \quad ,$$

$$\frac{d}{dz} y_3(z) = c(z)\omega y_2(z) \quad ,$$

$$\frac{d}{dz} y_4(z) = - c(z)\omega y_1(z) \quad . \tag{2.4.1}$$

and

$$y^t(z=0) = [a,0,0,0] \quad . \tag{2.4.2}$$

As shown in Fig. 2.4, 'x' is a forward and 'z' is a backward direction of integration. The independent variable for an analog computer is time 't'. The analog computer always integrates forward in time 't'. By setting t=z the equations (2.4.1) are integrated backwards in space.

The two adjoint systems have identical differential equations (from (2.3.26)),

$$\frac{d}{dx} \lambda_1(x) = - c(x)\omega\lambda_4(x) \ ,$$

$$\frac{d}{dx} \lambda_2(x) = c(x)\omega\lambda_3(x) \ ,$$

$$\frac{d}{dx} \lambda_3(x) = r(x)\lambda_1(x) - \omega\ell(x)\lambda_2(x) \ ,$$

$$\frac{d}{dx} \lambda_4(x) = r(x)\lambda_2(x) + \omega\ell(x)\lambda_1(x) \ . \qquad (2.4.3)$$

With the boundary conditions specified by

$$\lambda(x=0) = \lambda^\phi(0) = [-1,0,0,0]^t \qquad (2.4.4)$$

the solutions of (2.4.3) yield $\lambda^\phi(x)$ and with

$$\lambda(x=0) = \lambda^\Omega(0) = [0,-1,0,0]^t \qquad (2.4.5)$$

the solutions of (2.4.3) yield $\lambda^\Omega(x)$.

Equations (2.4.1) and (2.4.3) have the same form. With the transformation

$$P_1 = y_1(z) = \lambda_4(x) \quad , \quad .$$

$$P_2 = y_2(z) = \lambda_3(x) \quad ,$$

$$P_3 = y_3(z) = \lambda_2(x) \quad ,$$

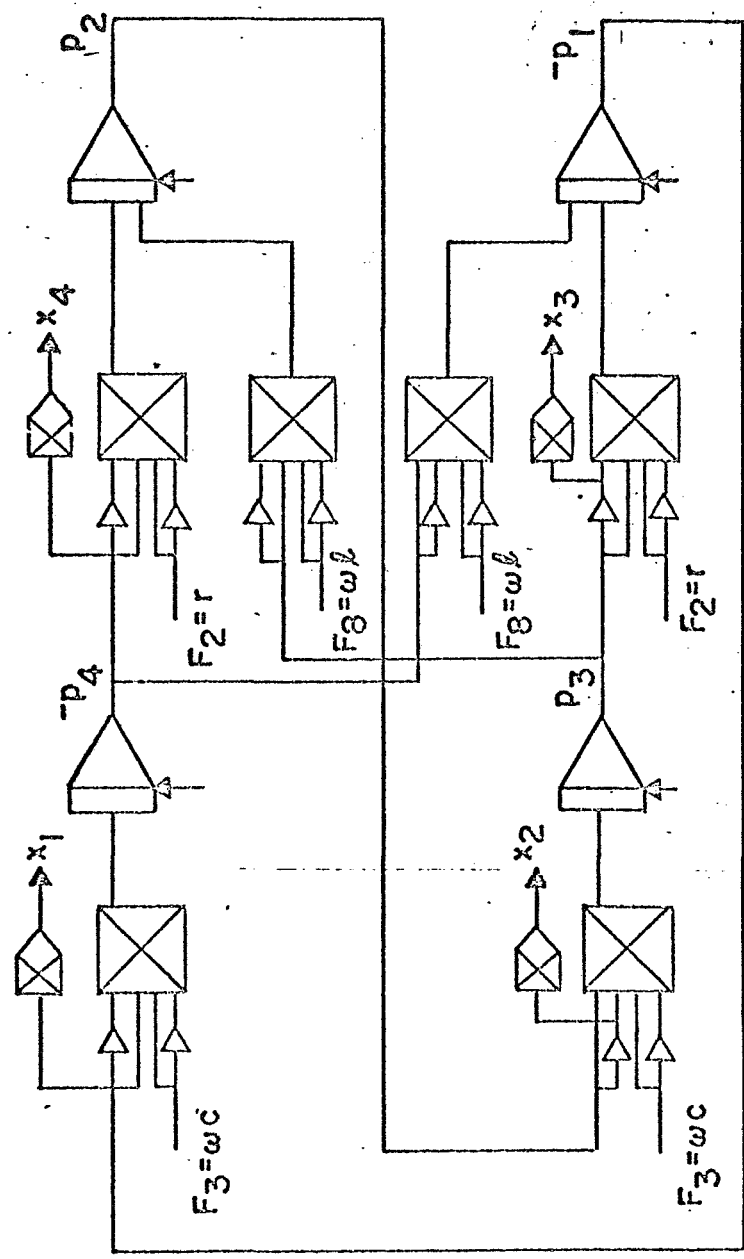$$P_4 = y_4(z) = \lambda_1(x) \quad . \tag{2.4.6}$$

and with the proper choice of space varying coefficients $r(x)$, $c(x)$, and $\ell(x)$ and the initial conditions, the same set of equations yield solutions for either $y(z)$, $\lambda^\phi(x)$ or $\lambda^\Omega(x)$.

The "B" matrix referred to in (3.31) turns out to be

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

The analog computer patching is given in Fig. 2.6.

The Hybrid Computer solution was attempted at first using the First Order Gradient Technique. The previous experience of the other investigators in the field of optimization indicated that the method would pose serious convergence problems. So was the case. An algorithm suitable for Hybrid Computational technique was developed, taking into consideration the second order variations. As described in Part B of this section, the method failed to improve convergence. It is now understood that the convergence is seriously affected by the truncation procedure resulting from the upper and lower bounds on the control. Part A in the following section describes the development and the

FIGURE 2.6

ANALOG SET-UP

$p_1 = y_1(z) = \lambda_4(x)$

$p_2 = y_2(z) = \lambda_3(x)$

$p_3 = y_3(z) = \lambda_2(x)$

$p_4 = y_4(z) = \lambda_1(x)$

INTEGRATOR

INVERTOR

MULTIPLIER

TRACK & STORE

procedural details of the Improved First Order Gradient Technique.
This technique successfully tackles the convergence problem and
yields "unique" distributions for the optimal control.

A.  Algorithm for the Improved First Order Gradient Technique:

The flow chart in Fig. 2.7 describes the hybrid program for
the first order estimation of the correction by the improved
gradient technique.  A more elaborate description is given below.

Block 1:  Preparatory Steps -- The input/output channels
of the DC are reset; the length of integration is specified; the
quantum of the x or z interval which results from discretiza-
tion of the space is calculated.  (The functions $r(x)$ and $c(x)$
are approximated by the staircase approximation.)  The upper and
lower limits on the control variables are specified and the
arbitrary initial profile of the control variables is assumed
and loaded into the memory.

Block 2:  Solving the System Equations on Hybrid Unit --
The DC sets the initial conditions for the integrators of the
analog computer as given by (2.4.2).  The initial values for the
integrators can be obtained either by (a) using the digital
computer to set a pot or (b) using DAC output lines.  The initial
values of the functions $r(z)$ and $c(z)$ are set up on the DAC.
The static test may be carried out at this time to check the
initial conditions.

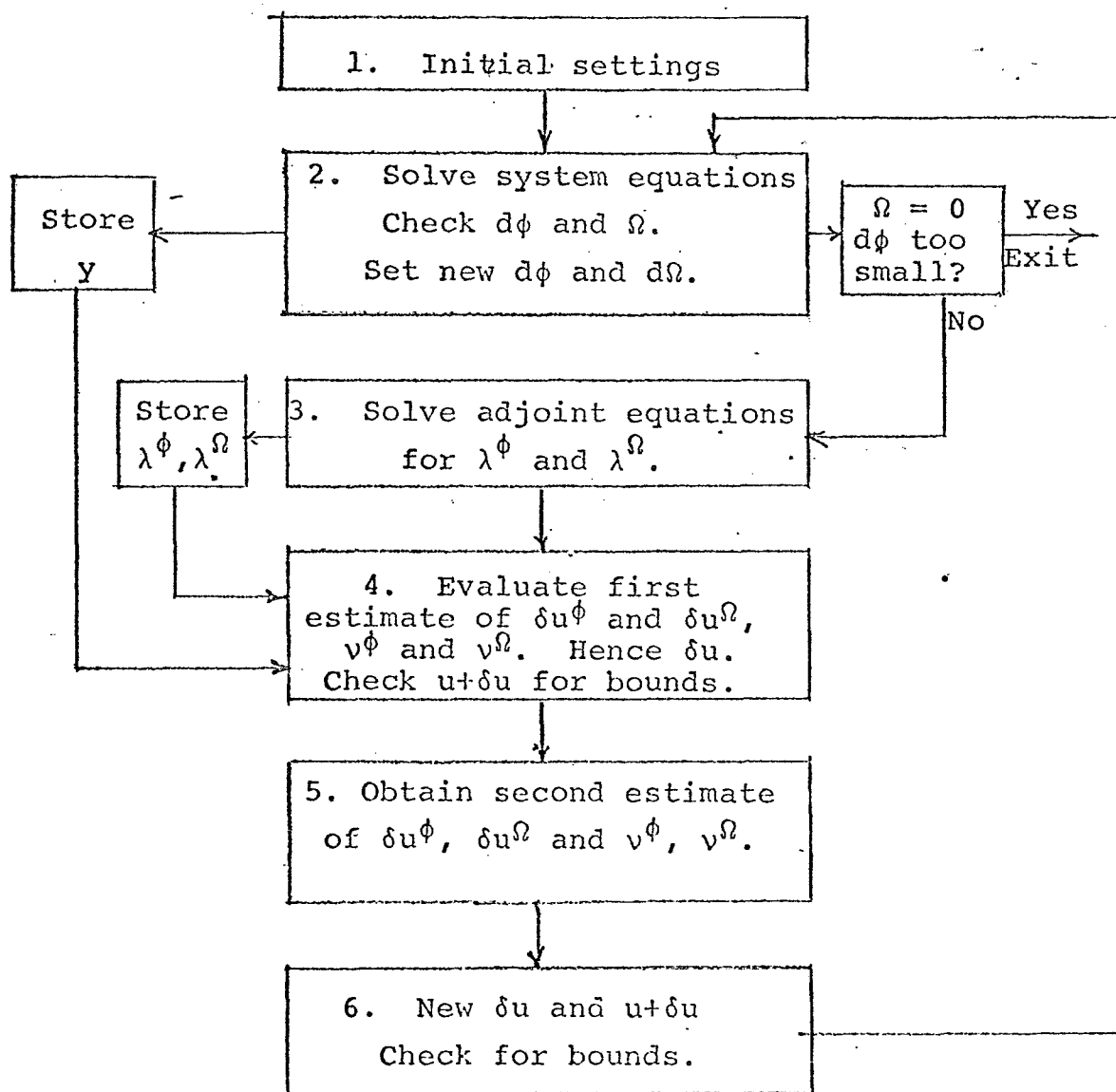The integration routine then follows.  The analog computer

```
┌─────────────────────────────────┐
│  1.   Initial settings          │
└─────────────────────────────────┘
```

```
┌────────┐      ┌──────────────────────────┐      ┌────────────┐   Yes
│ Store  │◄─────│ 2.   Solve system equations      │ Ω = 0      │──────►
│   y    │      │    Check dφ and Ω.               │ dφ too     │  Exit
│        │      │    Set new dφ and dΩ.            │ small?     │
└────────┘      └──────────────────────────┘      └────────────┘
                                                       │ No
```

```
┌────────┐   ┌──────────────────────────┐
│ Store  │◄──│ 3.   Solve adjoint equations │
│ λ^φ,λ^Ω│   │    for λ^φ and λ^Ω.          │
└────────┘   └──────────────────────────┘
```

```
┌──────────────────────────────────┐
│  4.   Evaluate first             │
│ estimate of δu^φ and δu^Ω,       │
│   ν^φ and ν^Ω.  Hence δu.        │
│ Check u+δu for bounds.           │
└──────────────────────────────────┘
```

```
┌──────────────────────────────────┐
│ 5. Obtain second estimate        │
│ of δu^φ, δu^Ω and ν^φ, ν^Ω.      │
└──────────────────────────────────┘
```

```
┌──────────────────────────────────┐
│  6.   New δu and u+δu            │
│    Check for bounds.             │
└──────────────────────────────────┘
```

FIGURE 2.7

COMPUTING ALGORITHM FOR IMPROVED HYBRID COMPUTING TECHNIQUE .

solves the system differential equations. The equations speci-
fied by (2.4.1) are integrated backwards in space. During
integration the AC receives from the digital computer the values
of the variable coefficients on the DAC and transmits back the
values of the system variables on the ADC.

The solutions obtained by integration are converted to
digital form and are stored in the memory of the digital computer.
The values of the criterion function and the residue for the
constraining equation are evaluated as,

$$\phi = y_1(z=L)$$

$$\Omega = y_2(z=L) \ .$$

The variation in the criterion function, $d\phi$, is chosen so as to
drive $y_1(x=0)$ towards the value of $y_1(L)$ such that the attenua-
tion approaches unity and $d\Omega$ is chosen so that $\Omega$ constraint is
rigorously satisfied by the next set of distributions, i.e.

$$d\Omega = -\Omega \qquad (or \ \Omega + d\Omega = 0)$$

Block 3: Solving the Adjoint Equations on the Hybrid Unit --
The operations are identical to the previous block except that
(a) the initial conditions are specified by (2.4.4) and (2.4.5)
for $\lambda^\phi$ and $\lambda^\Omega$ respectively, and (b) the adjoint equations (2.4.3)
are integrated forward in space so that $x=t$. Hence the control
distributions are $r(x)$ and $c(x)$. The same analog program that

is used for the system equations is used for the adjoint systems with the transformation of variables as given by (2.4.6).

Block 4: The First Estimate of $\delta u$ -- The equations (2.3.35), (2.3.36), (2.3.38), (2.3.39) and (2.3.37) yield the estimate of $\delta u$. (W is assumed to be an identity matrix.) Adding $\delta u$ to u one gets an estimate of the new control as $u+\delta u$. However, if u lies close to or is equal to the limiting values the new control $u+\delta u$ may exceed the limits. Under these circumstances $u+\delta u$ is confined to the limiting values wherever it exceeds the limits on the control variables. This amounts to the truncation of $\delta u$ so that $u+\delta u$ lies within the specified limits (see Fig. 2.9).

The estimated new control is monitored at this point to check if it exceeds the bounds and truncated if necessary. In the case of the unimproved gradient technique, the program branches back from here to block 2 and starts the new iteration loop.

It is observed that after the control variables reach the limiting values and start getting truncated, the subsequent iterations improve $\phi$ but cause $\Omega$ to diverge instead of converging to zero. It does not pay (in terms of convergence) to let $\Omega$ diverge too much. It becomes necessary to set a limit for $|\Omega|$ and monitor it at every iteration.

Whenever $\Omega$ diverges and exceeds the limit, only a $\Omega$ correction is applied during the iteration by assuming $\delta u^\phi = 0$ in equa-

tion (2.3.37). Thus

$$\delta u = \nu^{\Omega} \delta u^{\Omega}$$

during such iterations. This procedure drives $\Omega$ close to zero without any regard to the value of $\phi$. When $|\Omega|$ is driven sufficiently below the limiting value the imposed restriction '$\delta u^{\phi}=0$' can be removed and one can seek both $\phi$ and $\Omega$ corrections simultaneously.

So far we have not said anything about what values the elements of the weighting matrix $W$ should have. The matrix $W$ has to be positive so as to satisfy the strengthened Legendre necessary condition for the accessory problem. Normally $W$ is chosen to be an identity matrix. In such a case $H_u$ solely determines the shape of $\delta u$. $H_u$ is also the "sensitivity function" or the "influence function" for the improvement in the performance function with respect to a change in $u$. For the sake of clarity in the argument let us consider $u$ to be a one-dimensional vector. If we define $u_{opt}(x)$ as the optimal distribution of the control, then the variation needed to reach the optimal distribution from $u(x)$ in one step is $(u_{opt}-u)$. The ratio $(u_{opt}-u)/H_u$, which may be called "coefficient factor" is a good indicator of the uniformity in the system variation. If the coefficient factor is a constant then one can equate this constant to a step size $\nu$ so that

$$\delta u = \nu H_u = u_{opt} - u$$

and reach the optimum in one step. However, normally the coeffi-
cient factor varies over the range of x, implying thereby that
the control u(x) may be already close to the optimal profile in
the most sensitive regions and farther away in the least sensitive
regions. Hence it is impossible to reach the $u_{opt}$ in one step
unless one makes a fortuitous choice of factor $W^{-1}$. When u is
a two dimensional vector a matrix W(x) such as the one defined in
(2.4.7) can be used as a compensating factor. The matrix

$$W^{-1} = \begin{bmatrix} \dfrac{(r_M - r_m)}{L} x + r_m & 0 \\ 0 & c_M - \dfrac{(c_M - c_m)}{L} x \end{bmatrix} \quad (2.4.7)$$

has been found to be helpful in the present case. The choice
of W was governed by the sensitivity.

However, once $\delta u$ starts getting truncated at the boun-
daries, we face a different type of convergence problem. In
order to get a better understanding we will first consider a
simpler analogy.

Let $\phi = \phi(x_1, x_2)$ be the cost function of scalars $x_1$ and $x_2$.

Figure 2.8 shows the contours of the level lines for constant
$\phi$ in space $(x_1, x_2)$. The variables $x_1$ and $x_2$ are bounded. We have
to seek a minimum of $\phi$. Let $x^0 = (x_1^0, x_2^0)$ be an arbitrary
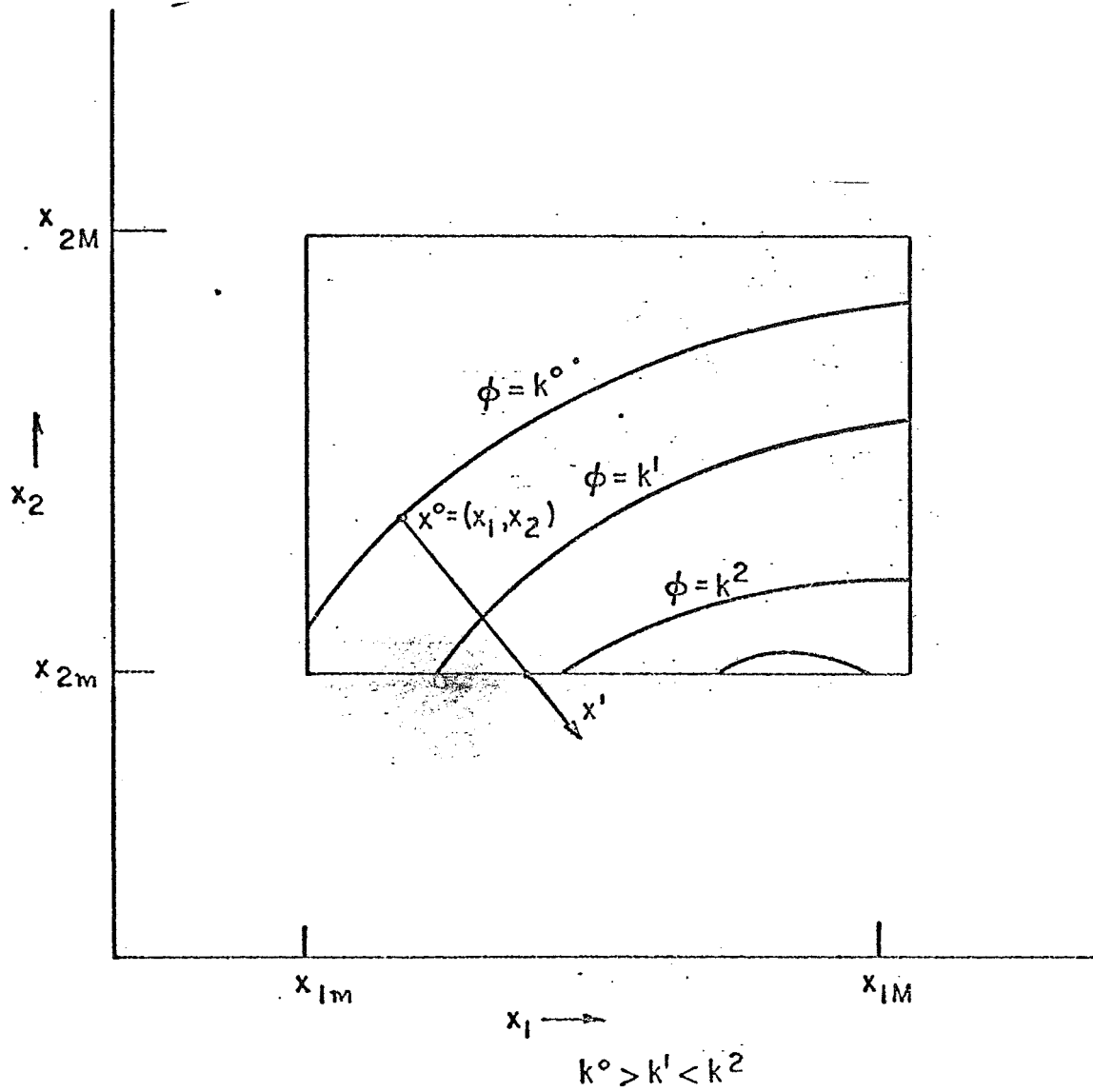
FIGURE 2.8

LEVEL LINES FOR ORDINARY MINIMIZATION PROBLEM

starting point. In the gradient technique one seeks to move in the direction of the negative gradient $-\nabla\phi$ which is normal to the level line $\phi=k^0$ at $x^0$. The step size is estimated from the desired improvement $d\phi$. If $x^0$ is close to the boundary of $x_1$ or $x_2$ the step in the negative gradient direction may go past the boundary as shown in Fig. 2.8. One has to 'truncate' the step at $x'$ which is a point on the boundary. It is apparent that from this point on, the step in the direction of the negative gradient will be truncated in the $x_2$ direction. The truncated step will yield much less improvement than the stipulated $d\phi$. This seriously affects the convergence.

It is obvious from the figure that the best direction to follow is

$$\begin{bmatrix} -\dfrac{\partial \phi}{\partial x_1} \\ \\ 0 \end{bmatrix}$$

i.e. to keep moving along the boundary $x_2 = x_{2m}$.

Block 5: Revised Estimate of $\delta u$ -- Let us consider the situation shown in Fig. 2.9 where u is a scalar function. A part of u lies on the boundary $u_m$ and a part lies on $u_M$. The variations $\delta u\phi$ and $\delta u^\Omega$ are the components obtained as described in Block 4. The variation $\delta u$ is the first estimate. However, .after truncation it reduces to $\delta u'$. It is apparent that a large section of $\delta u$ -- shown hatched -- was counted upon to make substantial contribution towards the variations $d\phi$ and $d\Omega$, but
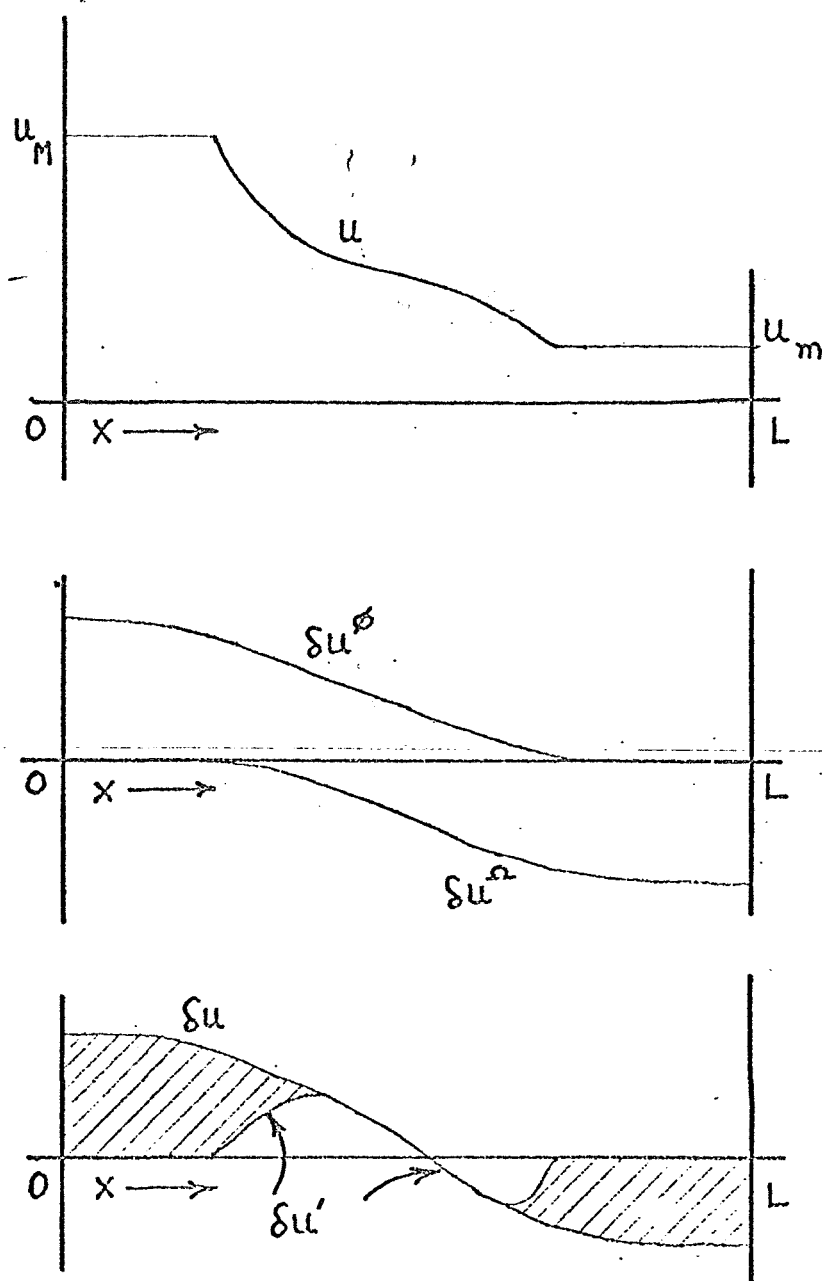
FIGURE 2.9

PLOTS OF $u$, $\delta u^\phi$, $\delta u^\Omega$ AND $\delta u'$ AS FUNCTION OF $x$

is now ineffective. The composition of $\delta u'$ in terms of $\delta u^\phi$ and $\delta u^\Omega$ cannot be estimated. Since $\delta u^\phi$ affects $d\Omega$, and $\delta u^\Omega$ affects $d\phi$, the corrections $d\phi$ and $d\Omega$ resulting from $\delta u'$ are not only small but are at times far different from the stipulated values.

This can be remedied, to a large extent, by giving due consideration to the effect of truncation in the revised estimate of $\delta u$. This is effected by using $|\delta u'|$ or $(\delta u')^2$ as a weighting factor. Thus we have

$$\delta u^{\phi'} = (\delta u')^2 \delta u^\phi$$

$$\delta u^{\Omega'} = (\delta u')^2 \delta u^\Omega \qquad\qquad (2.4.8)$$

Wherever the first estimate $\delta u$ gets truncated, $\delta u'$ is equal to zero (see Fig. 2.9). The new estimates $\delta u^{\phi'}$ and $\delta u^{\Omega'}$ will also be zero wherever the first estimate $\delta u$ is truncated. Thus, the second estimate of the step sizes $\nu^\phi$ and $\nu^\Omega$ is obtained by reshaping $\delta u^\phi$ and $\delta u^\Omega$ so that the second estimate of the variation $\delta u$ is confined, as far as possible, to the region where the possibility of the variation exists.

Revised values of $\nu^\phi$ and $\nu^\Omega$ may be obtained from (2.3.38) and (2.3.39) and the new estimate of $\delta u$ is given by

$$\delta u = \nu^\phi \delta u^{\phi'} + \nu^\Omega \delta u^{\Omega'}$$

Effectively we use a W factor, so that

$$
W^{-1} = \begin{bmatrix} (\delta r')^2 [\dfrac{(r_M - r_m)}{L} x + r_m'] & 0 \\[2em] 0 & (\delta c')^2 [c_M - \dfrac{(c_M - c_m)}{L} x] \end{bmatrix}
$$

$$(2.4.9)$$

where $\delta r'$ and $\delta c'$ are truncated first estimates from Block 4.

The last part of this operation is checking and truncating $u + \delta u$. Then the program goes back to Block 2 for the next iteration cycle.

B.  Second Order Gradient Technique

The Fig. 2.10 shows the flow diagram for the second order technique. To a large extent the first few steps for the improved first order technique are repeated for the second order gradient technique.

Block 1 through 3 are same as described in Part A of this section.

Block 4: The first estimate of $\delta u^\phi$ and $\delta u^\Omega$ is evaluated in accordance with (2.3.79) and (2.3.81). The unknown Lagrange multipliers $\nu^\phi$ and $\nu^\Omega$ are assumed to be zero.

Block 5: The system variations $\delta y_\phi$ and $\delta y_\Omega$ corresponding to the variations in control $\delta u^\phi$ and $\delta u^\Omega$ are obtained by defining

$$u^\phi = u + \delta u^\phi \quad \text{and} \quad u^\Omega = u + \delta u^\Omega .$$

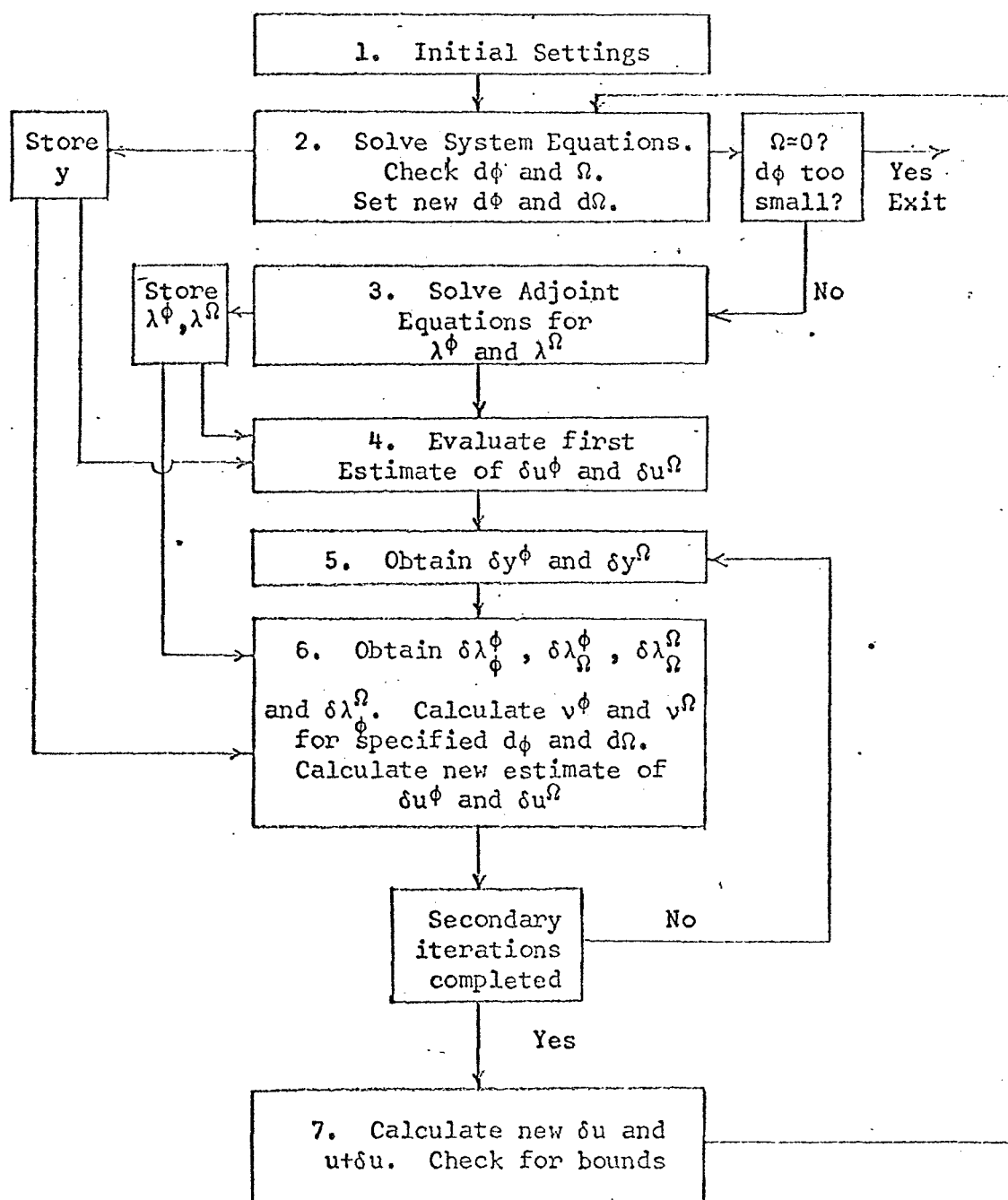With $u^\phi$ as a control variable the system equations are integrated

FIGURE 2.10

COMPUTING ALGORITHM FOR THE SECOND
ORDER GRADIENT TECHNIQUE

backwards in space as in Block 2. The solutions y obtained

during the operations of Block 2 are subtracted from these

functions to obtain

$$\delta y_\phi = y\big|_{u\phi} - y\big|_u \quad .$$

Similarly

$$\delta y_\Omega = y\big|_{u\Omega} - y\big|_u \quad .$$

Block 6: The adjoint system variations $\delta\lambda^\phi_\phi$, $\delta\lambda^\Omega_\phi$, $\delta\lambda^\phi_\Omega$, and

$\delta\lambda^\Omega_\Omega$ are obtained as follows. With $u^\phi$ as a control variable the

adjoint equations for $\lambda^\phi$ are integrated as in Block 3. The

solutions $\lambda^\phi$ obtained during the operations of Block 3 are sub-

tracted from these functions to obtain

$$\delta\lambda^\phi_\phi = \lambda^\phi\big|_{u\phi} - \lambda^\phi\big|_u \quad .$$

Similarly the other adjoint variations are obtained;

$$\delta\lambda^\Omega_\phi = \lambda^\Omega\big|_{u\phi} - \lambda^\phi\big|_u \quad ,$$

$$\delta\lambda^\phi_\Omega = \lambda^\phi\big|_{u\Omega} - \lambda^\phi\big|_u \quad ,$$

$$\delta\lambda^\Omega_\Omega = \lambda^\Omega\big|_{u\Omega} - \lambda^\Omega\big|_u \quad .$$

The estimates of $\nu^\phi$ and $\nu^\Omega$ are now updated with the help of

(2.3.75) and (2.3.76). The new values of $\nu^\phi$ and $\nu^\Omega$, and the

system and adjoint variations stored in the memory of the

digital computer are used in (2.3.79) and (2.3.81) to obtain

the new estimates of $\delta u^\phi$ and $\delta u^\Omega$. A cycle of the secondary

iteration loop is completed here. The program branches back to
Block 5 for the next cycle of the secondary iteration loop. It
was observed during the computer operations that within three to
four iterations a fairly good convergence is obtained for $v^\phi$,
$v^\Omega$, $\delta u^\phi$, and $\delta u^\Omega$. After a prespecified number of secondary
iterations the program exits from the secondary iteration loop
and procedes to Block 7.

Block 7: An estimate of the $\delta u$, required for the desired
change $d\phi$ and $d\Omega$, is obtained by using (2.3.68). This $\delta u$ is
added to u to obtain u+$\delta u$. This estimate of the control is
checked for the limits $u_M$ and $u_m$, and truncated wherever it tries
to cross the limits. Thus, a new estimate for the improved
control is obtained. A cycle of the main iteration loop is com-
pleted here. The program branches back to Block 2, for the next
iteration cycle of the main iteration loop.

During the Hybrid Computer execution of the program, it was
observed that the desired variations $d\phi$ and $d\Omega$, and the varia-
tions obtained as a result of the change in control "$\delta u$", obtained
as the second order estimate, are in very good agreement until
the control reaches the bounds. Once the control reaches the
boundary for any x, the estimates are erroneous and the solutions
stop converging. Thus, the second order technique does not help
to solve the convergence problem. Once the control reaches the
bounds it becomes necessary to use the improved first order
gradient technique.

## 2.5 Computer Solutions

The use of the Hybrid Computer was considered to be best suited for this problem due to the following reasons:

(i) The Analog Computer can solve the differential equations without discretization in 'x' space.

(ii) The Digital Computer with the help of D to A converter can generate arbitrary shapes of distributions and feed them to the Analog Computer to obtain the representation of a nonuniform transmission line.

(iii) The storage facility and the computational capability of the Digital Computer can be utilized to evaluate the estimates of $\delta u$.

Appendix A describes the Hybrid Computer operations. As a particular case of the oscillator problem we chose the following set of values for the numerical analysis.

The ratio of $r_M/r_m$ and $c_M/c_m$ is chosen to be 10. The limiting values are chosen to be

$$r_M = c_M = .8$$

$$r_m = c_m = .08$$

This choice is governed by the limitations of the dynamic range of the system. The ADC, DAC, and analog units cannot handle quantities larger than unity (10 volts), and for the values of the order of .0010 there is a serious noise problem. However,

a large spectrum of values can be handled by transforming the
independent variable (thus effectively changing the scale)
provided the dynamic range is not too large.

An exponential line such as

$$r(x) = re^{2kx} , \qquad r = r_m = 0.08$$

$$c(x) = Ce^{-2kx} , \qquad C = c_M = 0.8$$

$$\ell(x) = 0$$

is considered to be a good first estimate of u. Edson's [11]
graphs indicate that the length of the line required for 180
degree phase shift, with the above distribution, should be 17.3
units and the corresponding attenuation is 5. The calculations
show that the phase shift for line length of 17.3 units is 175
degrees and the attenuation is 4.7. When the exponentials are
generated on the Analog Computer and the system equations are
solved entirely on the analog unit, the length required for 180
degree phase shift is found to be 18.9 units and the correspond-
ing attenuation is 5.8. The calculations show that the phase
shift for line length of 18.9 units is 186 degrees and the
corresponding attenuation is 6.2. The discrepencies could be
attributed to the following factors.

(i) The Edson's graph leaves some room for ambiguity in
region of interest.

(ii) The analog multipliers are a bit noisy.

The quantum of the interval is chosen to be 1/10 unit. Thus, we have 189 discrete intervals for length of 18.9 units. The functions $r(x)$ and $\omega c(x)$ are represented by stepwise approximation. At the start of each interval the value of $r(x)$ or $\omega c(x)$ at that point is provided on DAC and held constant until the start of the next interval. The system variables are sampled on the ADC at the end of each interval. The sampled values are fed to the DC through ADC while the integration continues uninterrupted.

The first order unimproved gradient technique with W chosen as an identity matrix was tried first. Different distributions such as uniform distribution, ramp distribution, or exponential distribution were used as an initial guess. The problem of sensitivity was immediately felt since they did not converge to a single distribution.

The unimproved first order technique with W as an identity matrix indicated that with different initial guesses the iterations moved the distributions in the same general direction, but the sensitivity problems prevented them from converging to a single distribution. Also, the simultaneous convergence of $\phi$ and $\Omega$ was affected when the $\delta u$ estimates were truncated (see Appendix B). The algorithm described in Chapter 4 for the improved first order technique is an attempt to correct these defects. The method seems to work satisfactorily.

The unimproved first order technique with W as an identity

matrix is used until the improvement in $\phi$ becomes small. Then we branch to the method using (2.4.9) as weighting factor, which is the improved gradient technique.

The optimal distributions obtained from the computer are quite noisy. (The reasons are described in the next chapter.) Fig. 2.11 is the noisy output from the computer. Fig. 2.12 gives the filtered version. The rest of the figures presented here are the filtered versions of the computer output.

In order to check the dependence of the final distributions on the initial guess, two widely different sets of distributions are selected as an initial guess. In each case the length of the line is 14.

Case 1: The initial distributions are

$$r(x) = \omega c(x) = 0.325$$

The final distributions are given in Fig. 12.

Case 2: The initial distributions are

$$r(x) = 0.8 - (\frac{0.8-0.08}{L})x$$

$$\omega c(x) = 0.08 + (\frac{0.8-0.08}{L})x$$

The final distributions are given in Fig. 2.13.

The comparison of the results recorded in Fig. 2.12 and Fig. 2.13 shows that in both cases the distributions converged to the same set of final distributions. This indicates that the algorithm derived here is quite insensitive to the choice of the initial distributions.
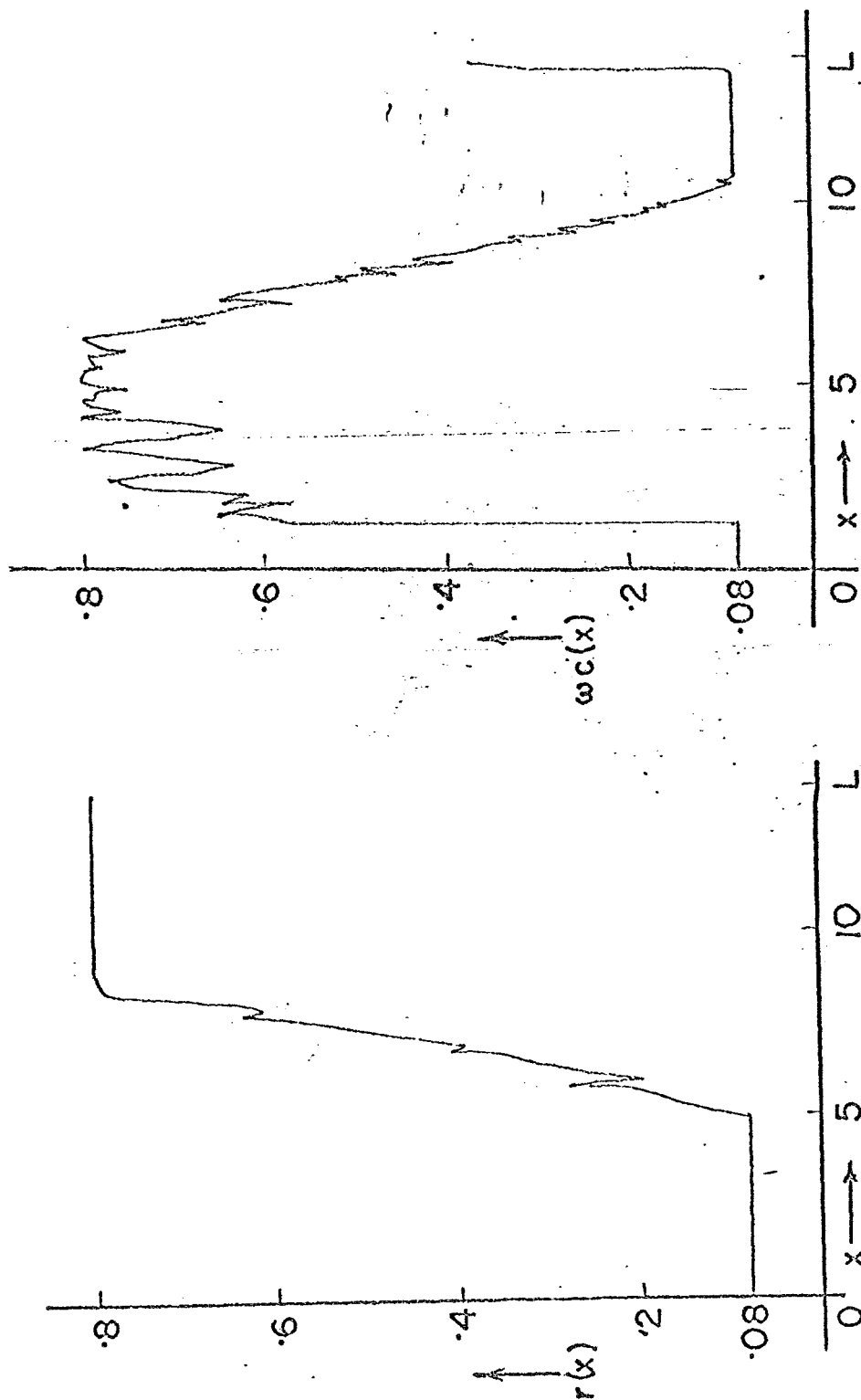
FIGURE 2.11

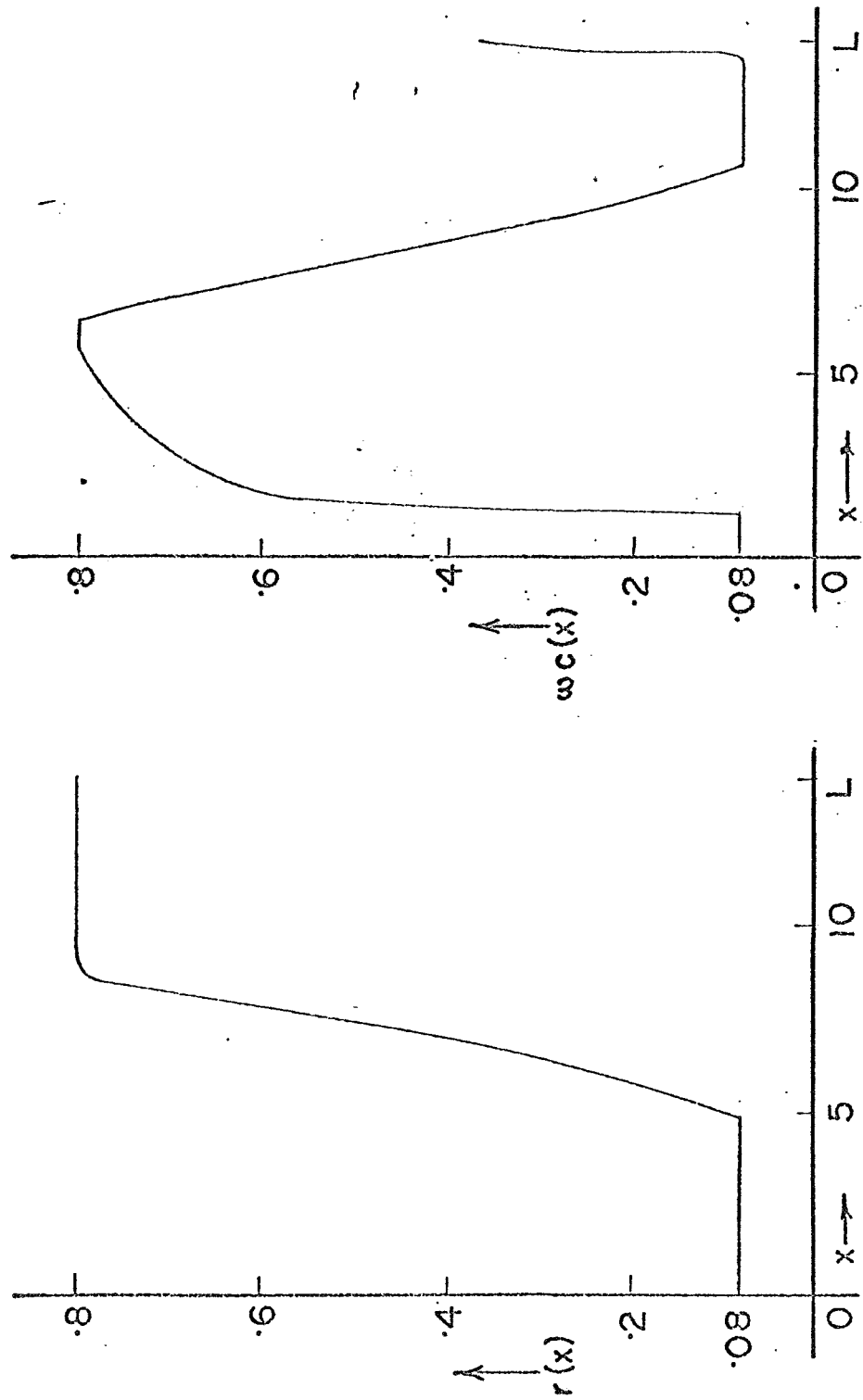Unfiltered Final Distributions, $L=14$ , $\ell(x)=0$

FIGURE 2.12    Filtered Final Distributions,  L=14,  $\ell(x)=0$

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a – r(x), Initial Distribution
b – r(x), Final Distribution

d – wc(x), Initial Distribution
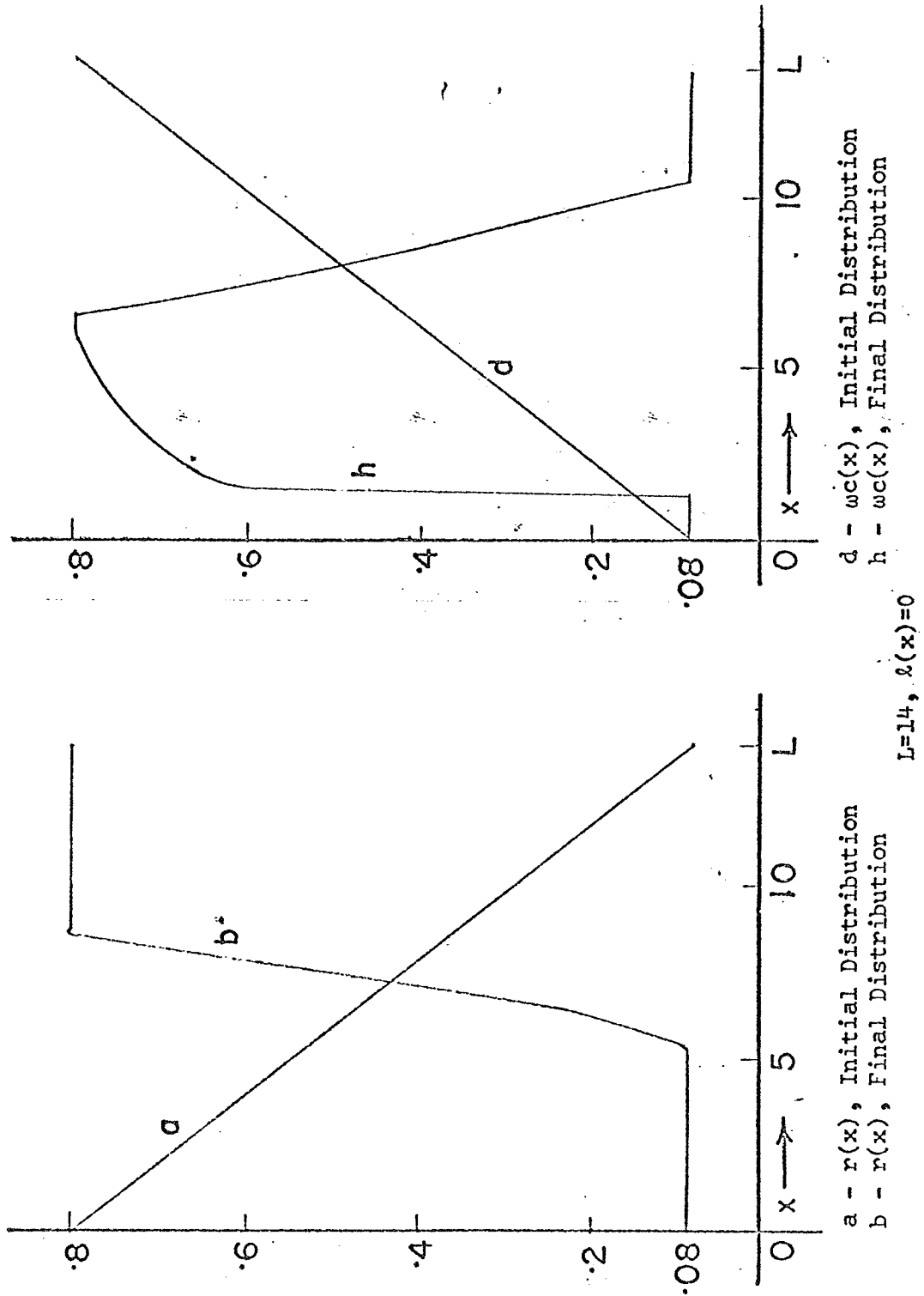h – wc(x), Final Distribution

L=14, ℓ(x)=0

FIGURE 2.13

INITIAL AND FINAL DISTRIBUTIONS, L=14

On the optimal switching curve in between the boundaries, $H_u$ should be identically zero. The observed values of $H_u$ at the start of the iteration and at the end differed by a factor of about 1000 indicating that the first distributions are very close to the optimum.

Figures 2.14 through 2.18 give the results for different assumed length 'L'. In each case the starting distributions are taken to be uniform and inductance $\ell(x)=0$. Table 2 summarizes these results. Figure 2.19 plots the optimum attenuation as a function of the line length.

For the second set of results we assumed different values for inductance $\ell(x)$. As stated before $\ell(x)$ was assumed to be non-controllable and constant.

Figures 2.20 through 2.22 present the optimal $r(x)$ and $c(x)$ for different $\ell(x)$. Table 3 summarizes these results.

Figure 2.23 shows the distributions obtained by using second order gradient technique.
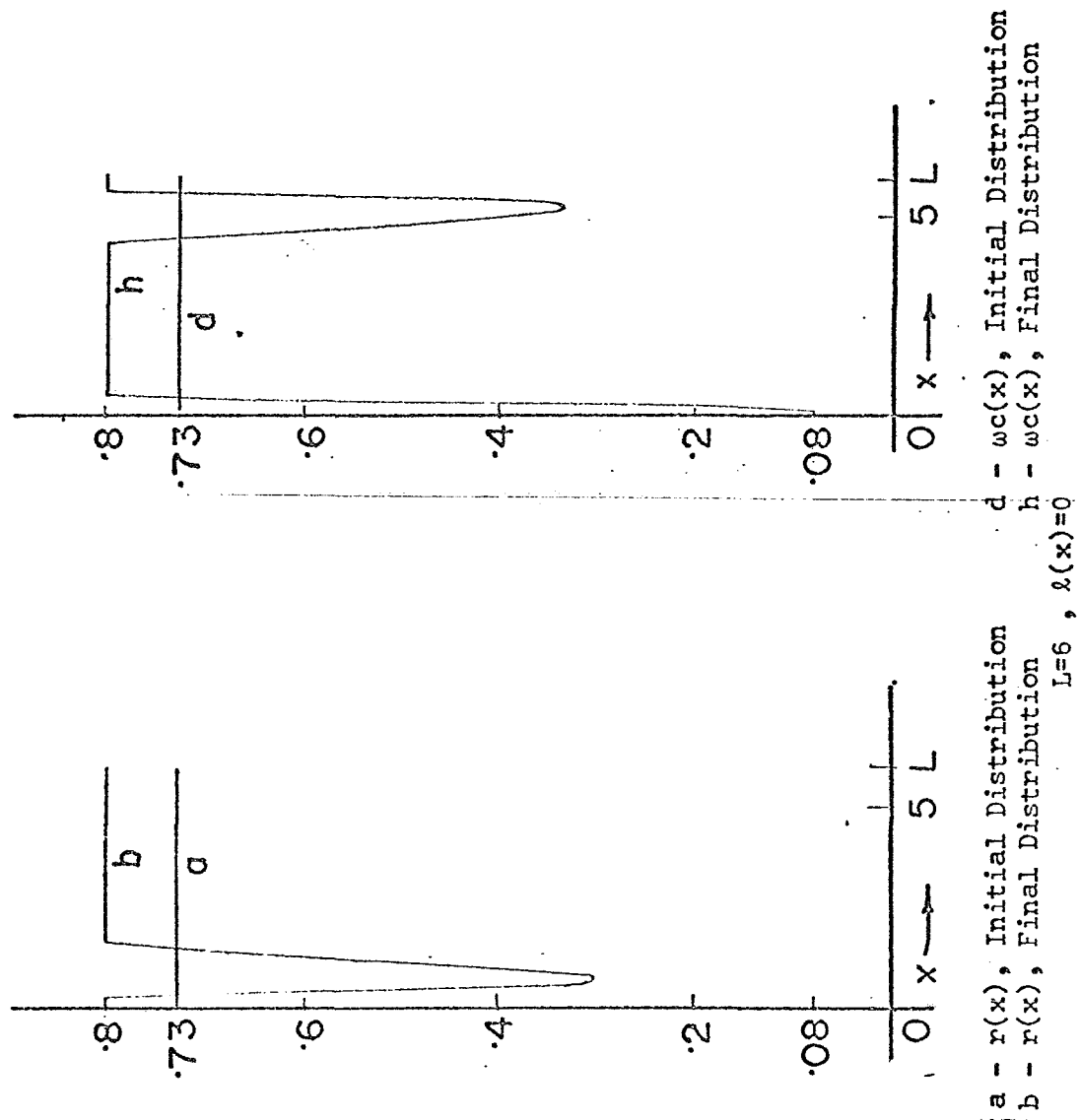
FIGURE 2.14

INITIAL AND FINAL DISTRIBUTIONS FOR DIFFERENT
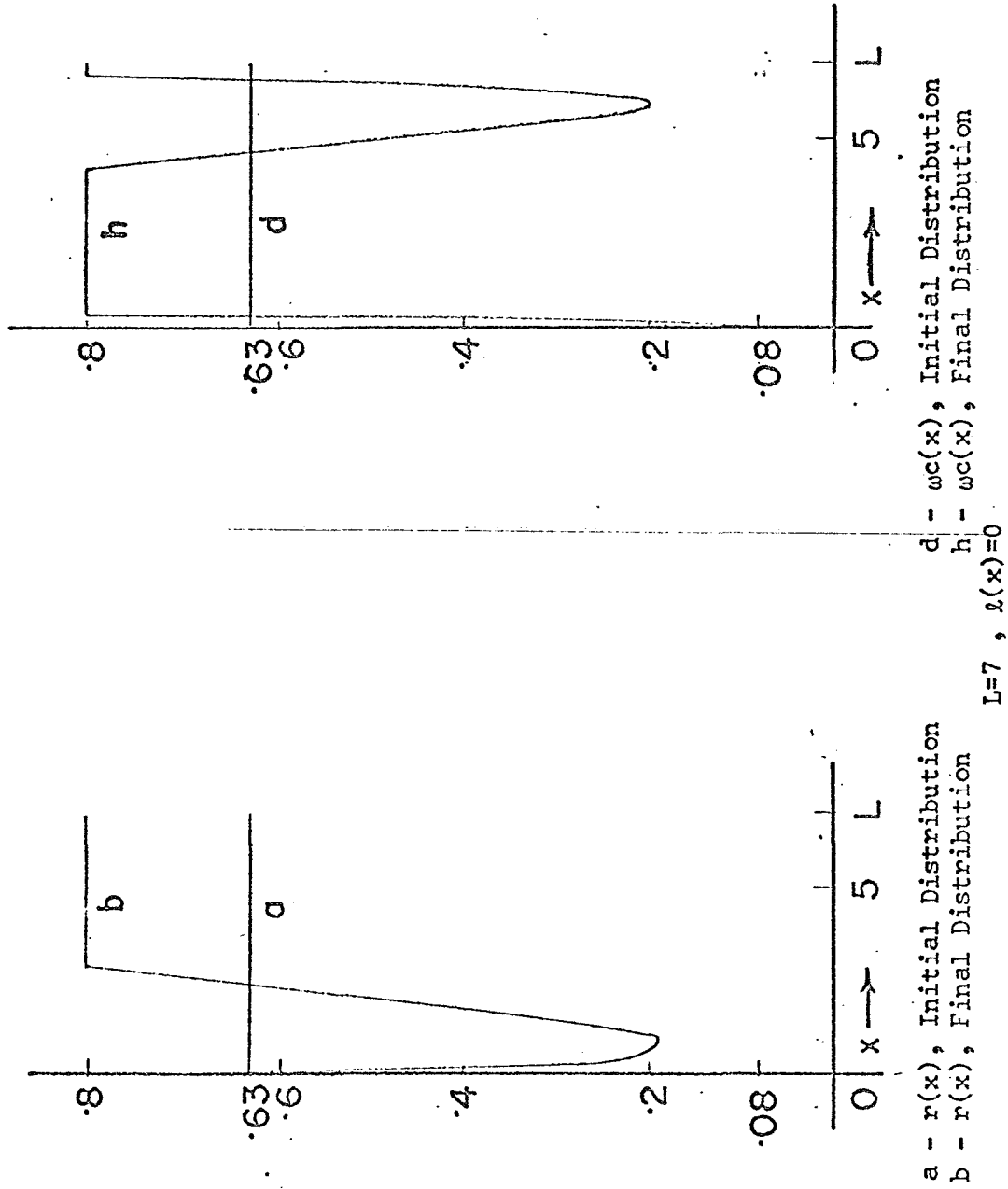ASSUMED INITIAL DISTRIBUTIONS, L=6, $\ell(x)=0$

a - r(x), Initial Distribution
b - r(x), Final Distribution
d - wc(x), Initial Distribution
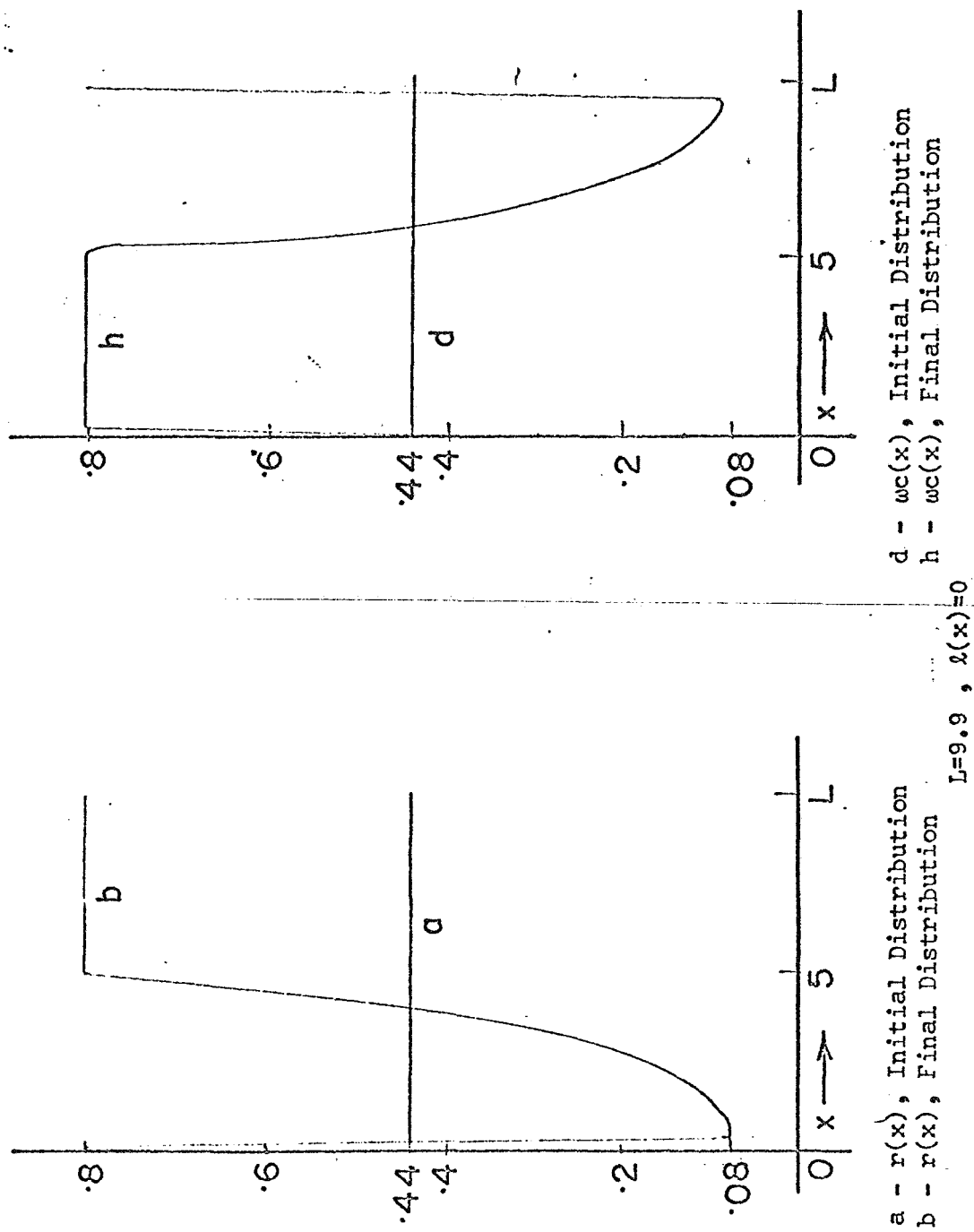h - wc(x), Final Distribution

L=6, $\ell(x)=0$

FIGURE 2.15

INITIAL AND FINAL DISTRIBUTIONS FOR DIFFERENT
ASSUMED INITIAL DISTRIBUTIONS, L=7, $\ell(x)=0$

FIGURE 2.16

INITIAL AND FINAL DISTRIBUTIONS FOR DIFFERENT.
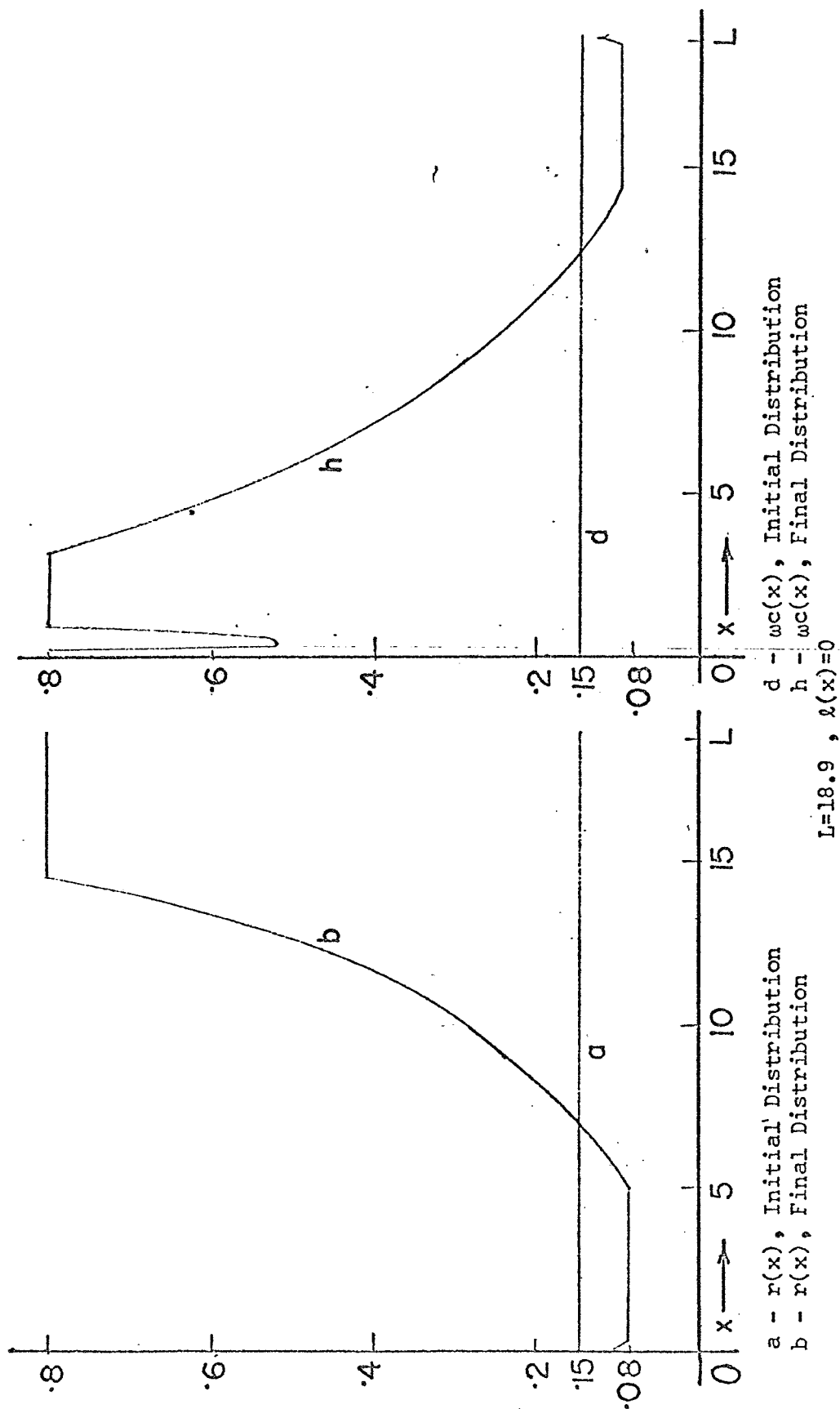ASSUMED INITIAL DISTRIBUTIONS, L=9.9, $\ell(x)=0$

FIGURE 2.17

INITIAL AND FINAL DISTRIBUTIONS FOR DIFFERENT
ASSUMED INITIAL DISTRIBUTIONS, L=17, $\ell(x)=0$.

a – r(x), Initial Distribution
b – r(x), Final Distribution

d – $\omega c(x)$, Initial Distribution
h – $\omega c(x)$, Final Distribution

L=18.9 , $\ell(x)=0$

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a — $r(x)$ and $\omega c(x)$, Initial Distribution
b — $r(x)$, Final Distribution
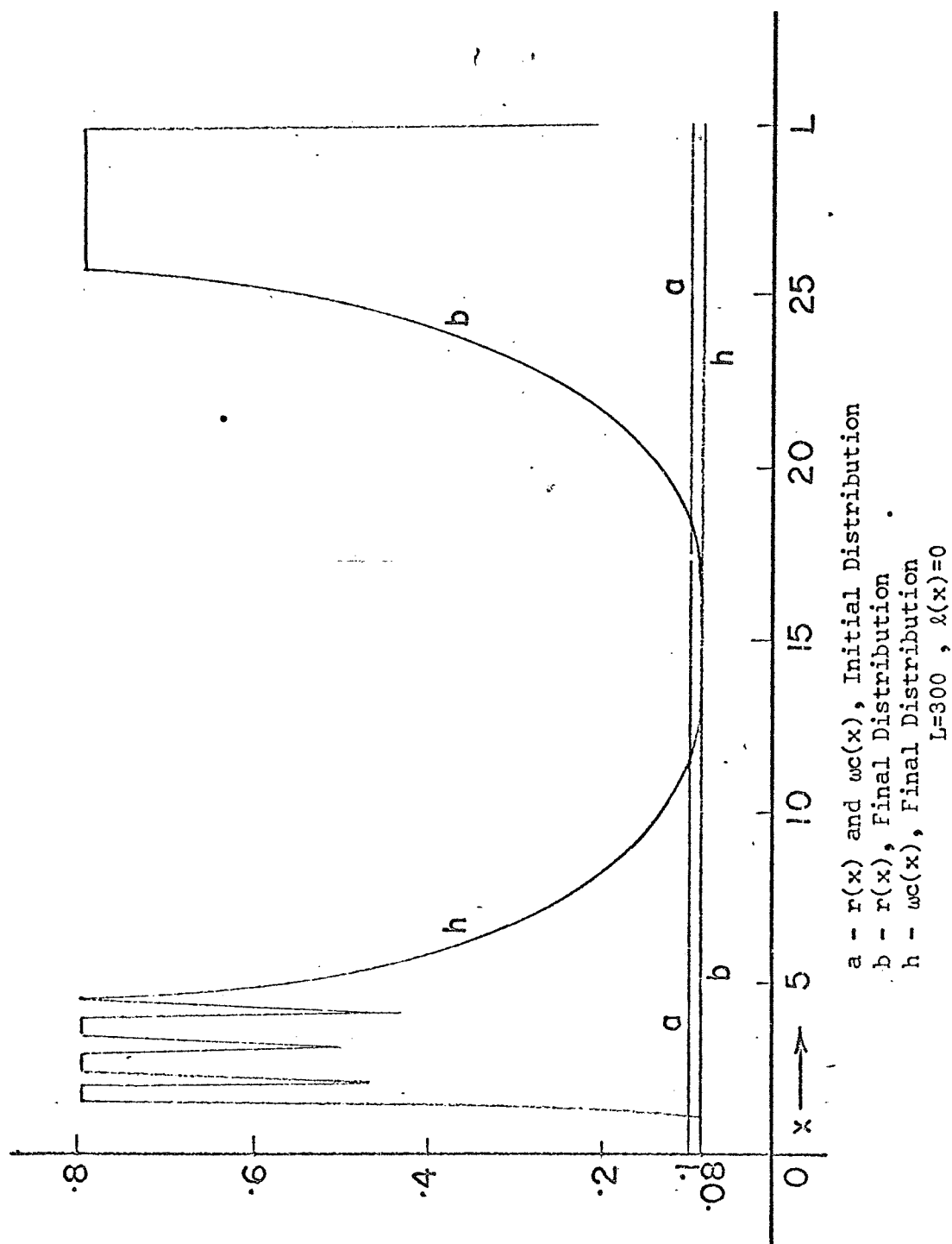h — $\omega c(x)$, Final Distribution
$L=300$ , $\ell(x)=0$

FIGURE 2.18

INITIAL AND FINAL DISTRIBUTIONS FOR DIFFERENT
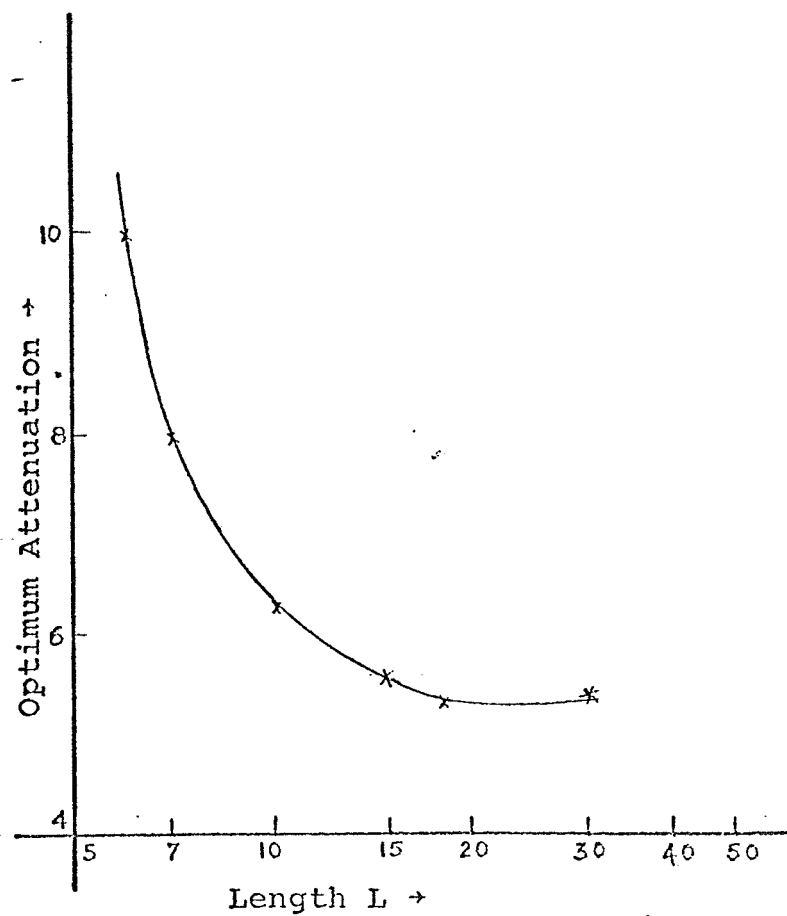ASSUMED INITIAL DISTRIBUTIONS, $L=18$, $\ell(x)=0$

FIGURE 2.19

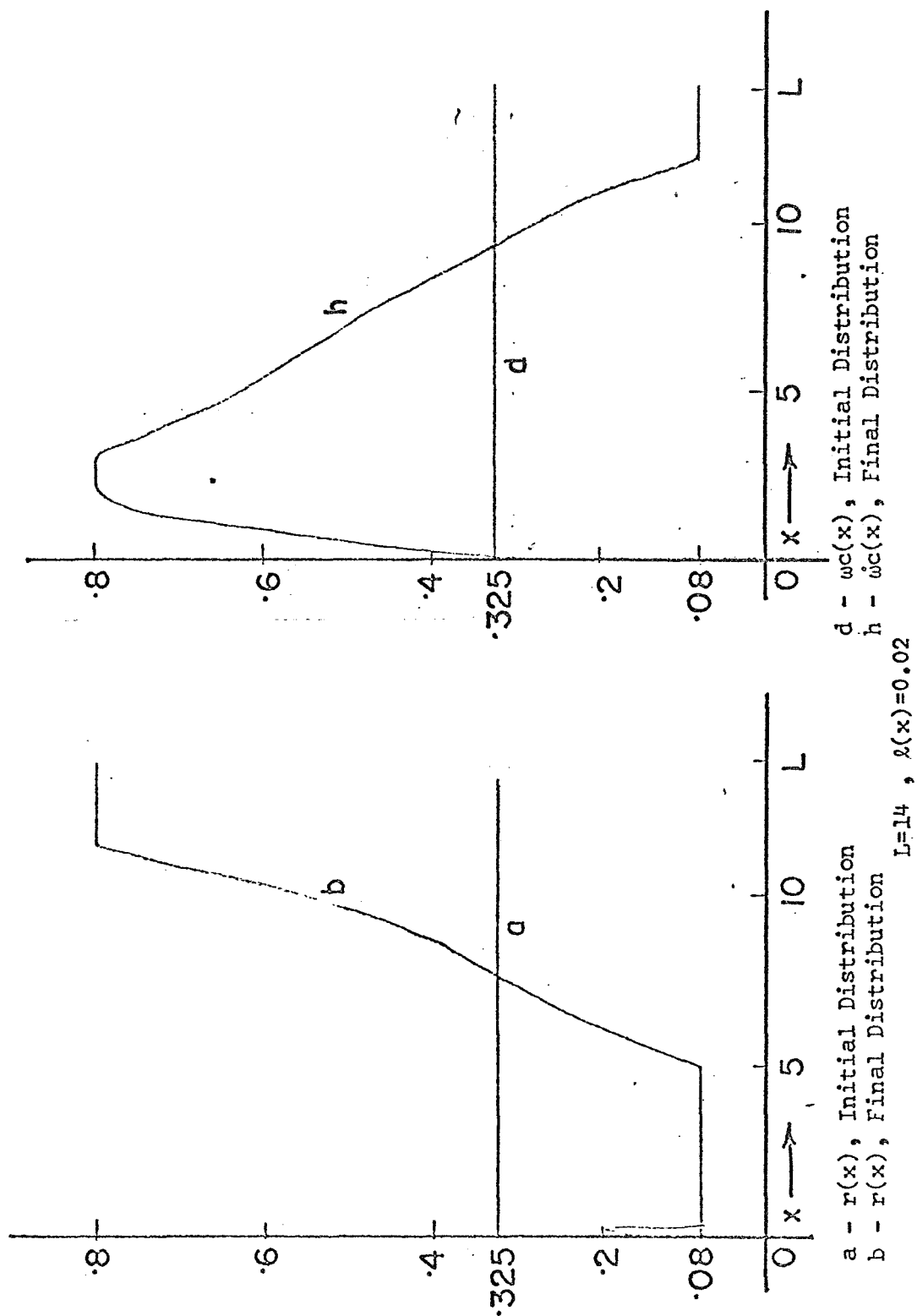OPTIMUM ATTENUATION AS A FUNCTION OF THE
TOTAL LENGTH OF A LINE

FIGURE 2.20

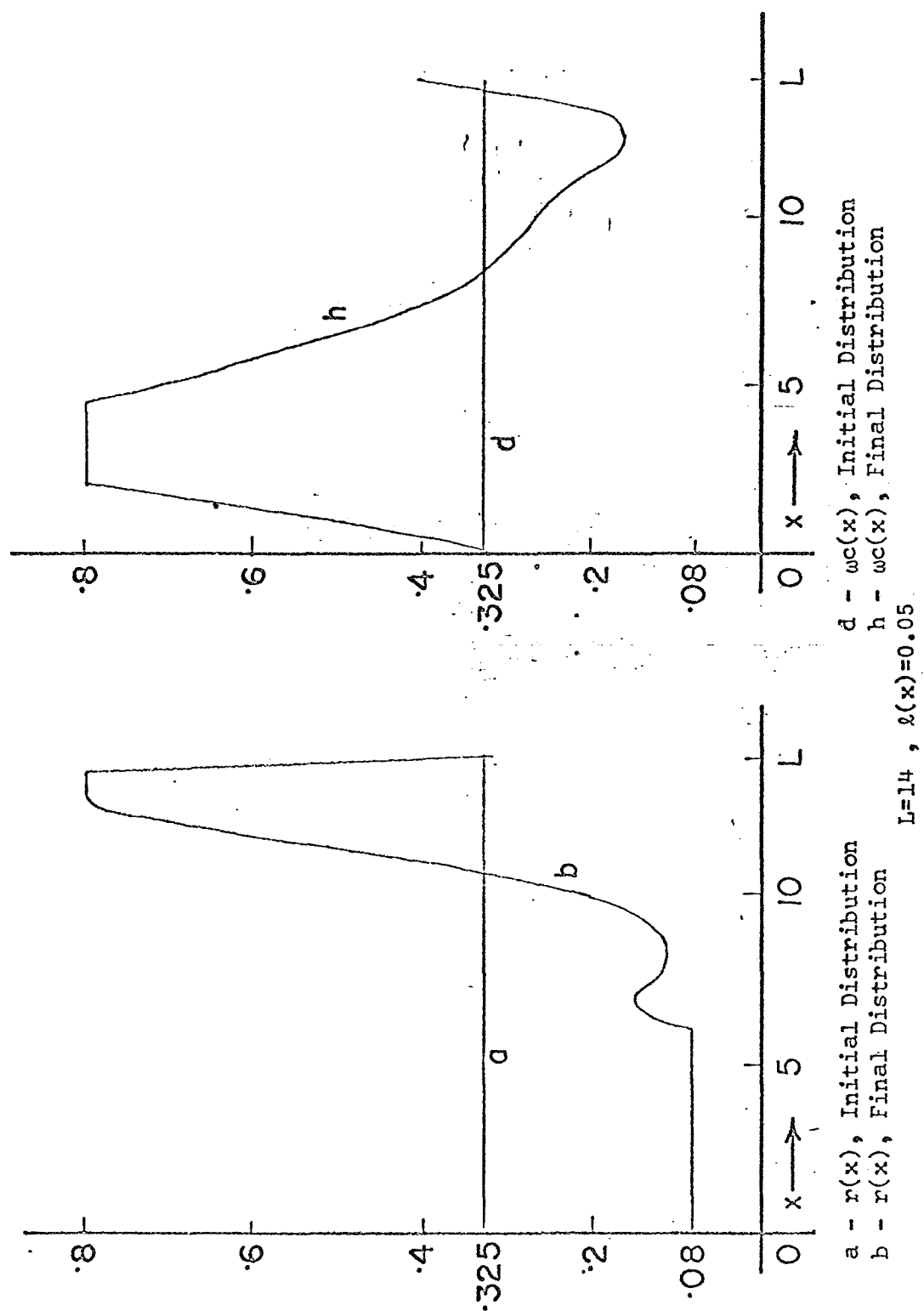INITIAL AND FINAL DISTRIBUTIONS FOR
ASSUMED CONSTANT VALUE OF $\ell(x)$
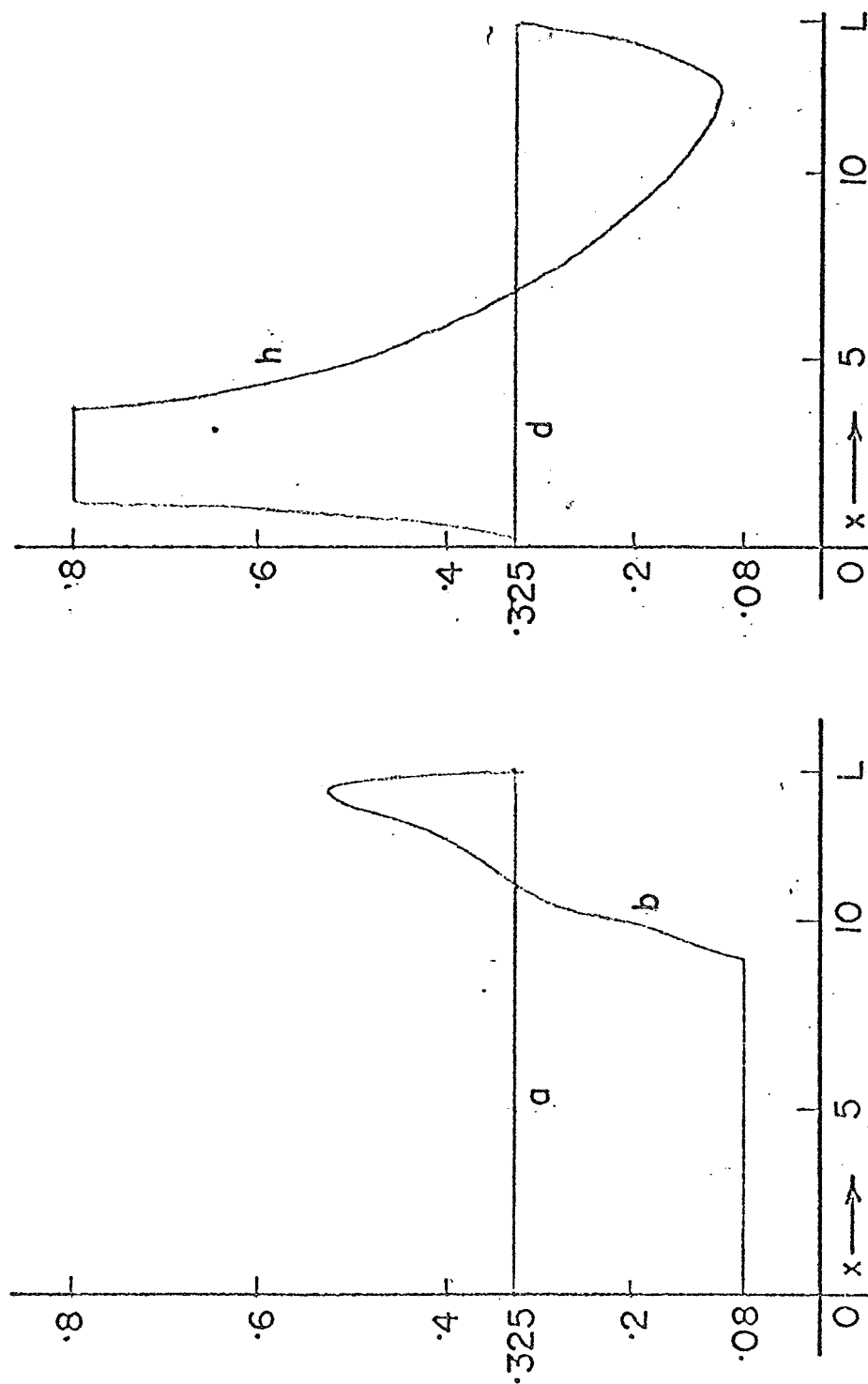
a – r(x), Initial Distribution
b – r(x), Final Distribution

d – ωc(x), Initial Distribution
h – ωc(x), Final Distribution

$L=14$ , $\ell(x)=0.05$

FIGURE 2.21

INITIAL AND FINAL DISTRIBUTIONS FOR
ASSUMED CONSTANT VALUE OF $\ell(x)$

a – r(x), Initial Distribution
b – r(x), Final Distribution

d – uc(x), Initial Distribution
h – uc(x), Final Distribution

L=14 ,$\ell(x)$=0.1
FIGURE 2.22

INITIAL AND FINAL DISTRIBUTIONS FOR
ASSUMED CONSTANT VALUE OF $\ell(x)$

TABLE 2

THE INITIAL UNIFORM DISTRIBUTIONS FOR 180 DEGREE PHASE SHIFT
AND THE CORRESPONDING 'OPTIMUM' ATTENUATION FOR
VARIOUS LENGTHS OF THE LINE

| Total length of line 'L' | Uniform distribution for 180 degree phase shift, $\omega c(x) = r(x)$ | Optimum attenuation with nonuniform distributions |
|---|---|---|
| 6.0 | .73 | 10 |
| 7.0 | .63 | 8 |
| 9.9 | .44 | 6.3 |
| 14.0 | .32 | 5.6 |
| 18.9 | .235 | 5.3 |
| 30.0 | .15 | 5.4 |

TABLE 3

OPTIMUM ATTENUATION FOR DIFFERENT VALUES OF $\omega\ell(x)$
TOTAL LENGTH OF A LINE 'L' = 14.0

| Uniform line inductance $\omega\ell(x)$ | Optimum attenuation with nonuniform $r(x)$ and $c(x)$ |
|---|---|
| 0 | 5.6 |
| .02 | 4.2 |
| .05 | 2.15 |
| .10 | 1.15 |

In each case $0.8 \leq r(x)$, $\omega c(x) \leq 0.08$.
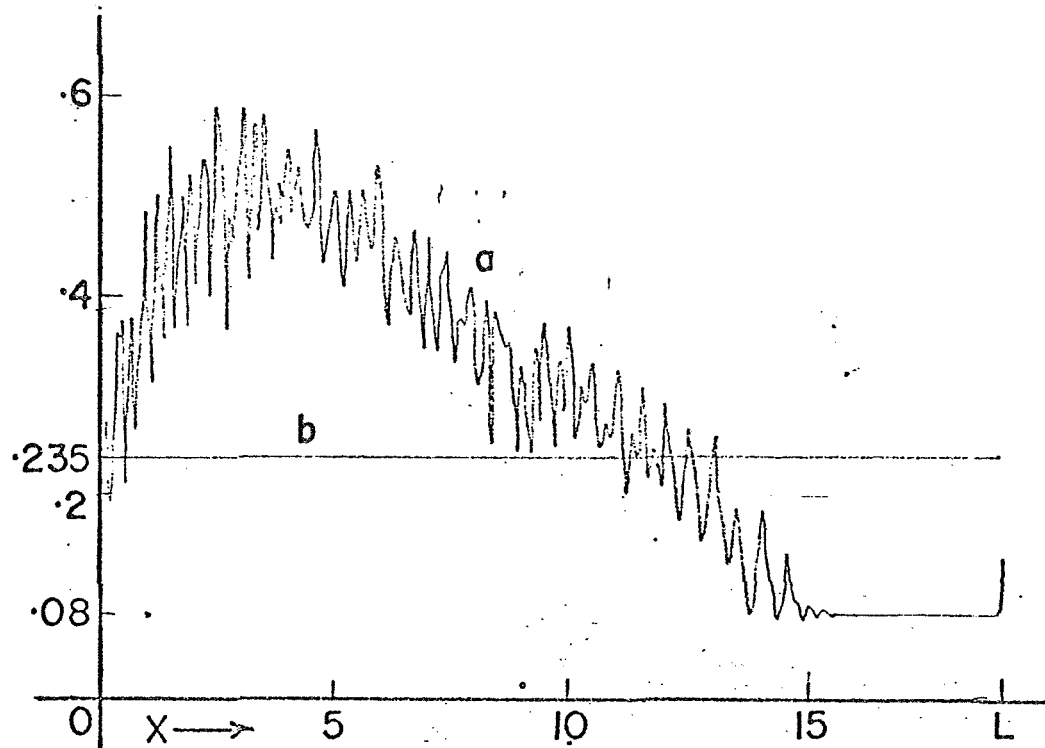
a-ωc(x), Final Distribution;b-ωc(x), Initial Distribution
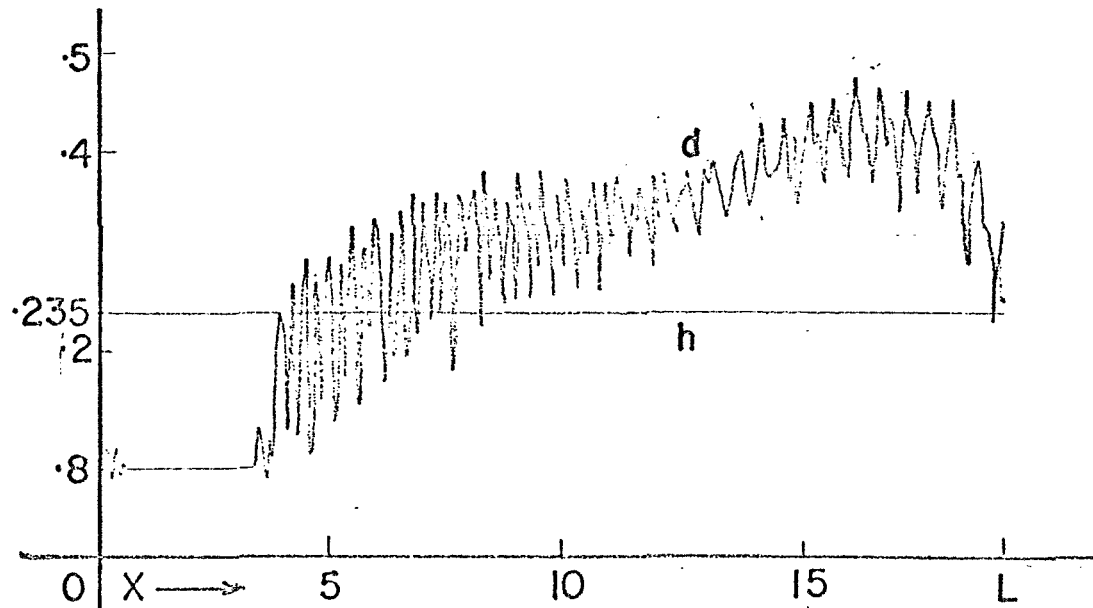
d-r(x), Final Distribution; h-r(x), Initial Distribution

Second Order Gradient Technique
L=18.9 , ℓ(x)=0

FIGURE 2.23

DISTRIBUTIONS OBTAINED BY USING SECOND
ORDER GRADIENT TECHNIQUE

CHAPTER 3

3.1        Notched Filter Synthesis

Many situations occur in the design of an electronic system
in which it is desirable to have frequency selective amplifica-
tion, i.e. amplify the inputs lying in a narrow band of fre-
quencies and reject all others. The selectivity is accomplished
by combining an amplifying element with a frequency selective
filter element. The amplifying element provides amplification
for a wide spectrum of frequencies and the filter element provides
the frequency discrimination. One such arrangement employing a
negative feedback loop is shown in Fig. 3.1.



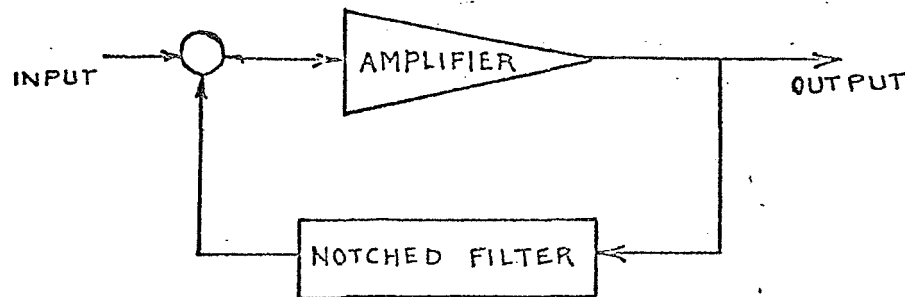FIGURE 3.1

BLOCK DIAGRAM FOR FREQUENCY SELECTIVE AMPLIFICATION

The overall gain of the system is given by

$$G_s = -A/(1 + GA)$$

where -A is the gain of the amplifier and G is the transfer func-
tion of the feedback network. At a null frequency $\omega_0$ transfer
ratio G becomes zero, thus effectively eliminating the feedback.

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The gain $G_s$ of the system becomes $-A$. As one moves away from $\omega_0$, G approaches unity and for very large values of A the gain $G_s$ drops to minus unity.

A desired transmission characteristic for such a feedback network could be as shown in Fig. 3.2
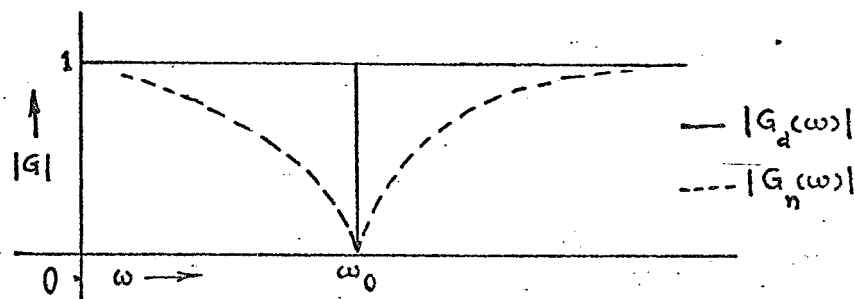


FIGURE 3.2

FREQUENCY CHARACTERISTICS OF A NOTCHED FILTER

The transmittance curve $G_d(\omega)$ shows the desired notched filter characteristics.

$$|G_d(\omega)| = 1 \qquad \text{for} \quad = \omega_0 \quad ,$$

$$|G_d(\omega)| = 0 \qquad \text{for } \omega = \omega_0 \quad . \qquad (3.1.1)$$

A more realistic and attainable characteristic is represented by $|G_n(\omega)|$ (Fig. 3.2). A physically realizable filter with characteristics such as $|G_n(\omega)|$ can be constructed [18] from resistive and capacitive components. A distributed parameter configuration with frequency selective null characteristics is shown in Fig. 3.3.
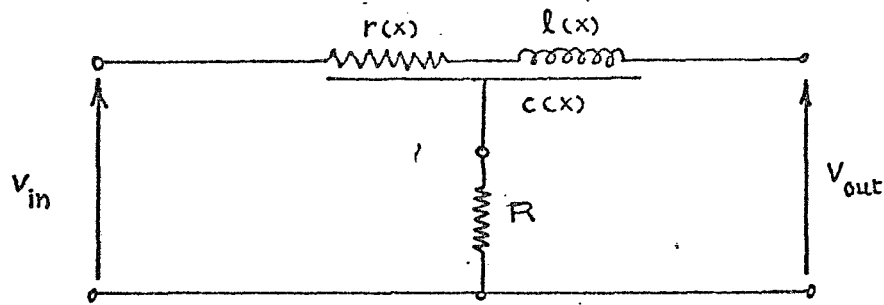
FIGURE 3.3

A DISTRIBUTED PARAMETER CIRCUIT FOR NOTCHED FILTER

The distributed part of the circuit is identical to the rc

filter shown in Fig. 1.1. A lumped resistance R is added in

series with the distributed capacitance c(x). Again, induc-

tance $\ell(x)$ is assumed to be a noncontrollable parameter and

only has significance at very high frequencies. Fuller and

Castro [18] have assumed a uniform rc distribution. In order

to have a null frequency at $\omega_0$, the parameter values for such

a distributed rc circuit are

$$\omega_0 r(x) c(x) L^2 = 11.12$$

and

$$R = 0.0563 \, r(x)L$$

If the distributions r(x) and c(x) are not restricted to

uniform distributions but are allowed to take a free form, it

should be possible to improve the performance of the filter. In

the sequel we have kept the form of the distributions completely

free except for the upper and lower bounds resulting from the

practical constraints of realizability and tried to obtain the

distributions of parameters $r(x)$ and $c(x)$ that yield the best

approximation to the frequency characteristics $|G_d(\omega)|$ as in

Fig. 3.2.

3.2     Notched Filter Problem Formulation

The problem formulation remains essentially the same as in

Section 1.2.  Given a source voltage $\alpha_{in}(\omega)$ over the frequency

interval $(-\infty, \infty)$, find the distributions $r(x)$ and $c(x)$ which

yield the output voltage $\alpha_{out}(\omega)$ such that the frequency response

characteristics $|G(\omega)|$, defined as

$$|G(\omega)| = \frac{\alpha_{out}(\omega)}{\alpha_{in}(\omega)}$$
(3.2.1)

is the best approximations to the desired response $|G_d(\omega)|$.  The

voltages $\alpha_{out}(\omega)$ and $\alpha_{in}(\omega)$ are related to the variables $V_1$, $V_2$,

$I_1$, and $I_2$ which describe the voltage and current relationships

along the distributed line.  These are

$$V_{1in}(\omega) = V_1(0,\omega) + RI_1(0,\omega) \quad ,$$

$$V_{2in}(\omega) = V_2(0,\omega) + RI_2(0,\omega) \quad ,$$

$$V_{1out}(\omega) = V_1(L,\omega) + RI_1(L,\omega) \quad ,$$

$$V_{2out}(\omega) = V_2(L,\omega) + RI_2(L,\omega) \quad .$$
(3.2.2)

The development from Equation (1.2.2) to (1.2.15) follows

along the identical lines.  The state variables, control variables

and the state equations depicting the behavior of the distributed

line remain the same.  However, the addition of lumped resistance

R alters the form of the criterion functional.  The problem can

be specified as

$$\frac{d}{dx} y = Ay = f(u,y) \qquad (3.2.3)$$

with A defined by (1.2.10a),

$$u^t = [r(x), c(x)]$$

and

$$y^t(x=L) = [a,0,0,0] , \qquad (3.2.4)$$

find u such that

$$J = \min_{u} \phi , \qquad (3.2.5)$$

where

$$\phi = \int_{-\infty}^{\infty} F(y(0,\omega)) \, d\omega \qquad (3.2.6)$$

and

$$F(y(0,\omega)) = \left| |G_d(\omega)|^2 - |G(\omega)|^2 \right| , \qquad (3.2.7)$$

subject to the constraints

$$r_m \le r(x) \le r_M$$

and

$$c_m \le c(x) \le c_M . \qquad (3.2.8)$$

From the definition (3.2.1) and relationships (3.2.2) we can express $|G(\omega)|^2$ as a function of R and the terminal values of the state variables $V_1$, $V_2$, $I_1$, and $I_2$.

$$|G(\omega)|^2 = \frac{(V_1(L) + I_1(0)R)^2 + (V_2(L) + I_2(0)R)^2}{(V_1(0) + I_1(0)R)^2 + (V_2(0) + I_2(0)R)^2} .$$

Substituting for the boundary conditions from (3.2.4) we obtain

$$|G(\omega)|^2 = \frac{(a + y_3(0)R)^2 + (y_4(0)R)^2}{(y_1(0) + y_3(0)R)^2 + (y_2(0) + y_4(0)R)^2}$$

$$(3.2.9)$$

In order to make the problem tractable with the help of computers the limits on $\omega$ are set as $\omega_m \leq \omega \leq \omega_M$ and the criterion function is discretized in $\omega$ with $\Delta$ as a quantum for discretization. (See Fig. 3.4)
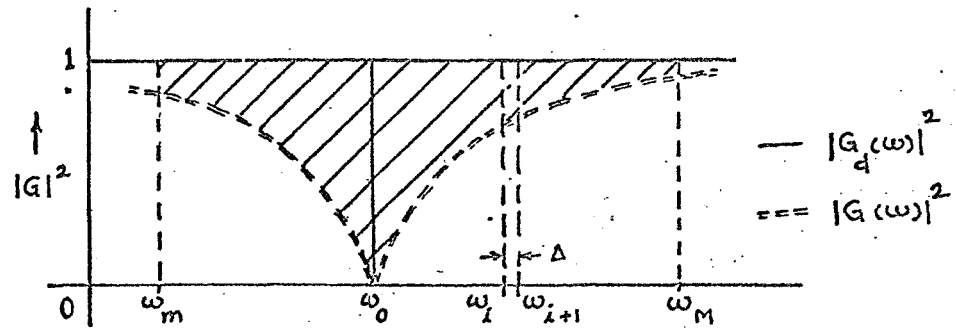


FIGURE 3.4

DISCRETIZATION OF THE FREQUENCY CHARACTERISTICS OF A NOTCHED FILTER

With these modifications the expressions (3.2.6) and (3.2.7) reduce to

$$\phi = \Delta \sum_{\omega_i} F(y(0,\omega_i)) ,$$

$$= \Delta \sum_{\omega_i} \left| |G_d(\omega_i)|^2 - |G(\omega_i)|^2 \right| , \qquad (3.2.10)$$

where i takes all the integral values from $-m_1$ to $m_2$. The minimization of $\phi$ implies the minimization of the area shown hatched in Fig. 3.4. However, since the point discontinuity in $G_d(\omega)$ at $\omega_0$ does not affect the total hatched area, the criterion

functional $\phi$ cannot differentiate between the following "desired" characteristics.

(i) $G_d(\omega) = 1$     at $\omega = \omega_0$ ,

        $= 0$    ? .at $\omega = \omega_0$ ;

and

(ii) $G_d(\omega) = 1$     for all $\omega$ .

Under these circumstances the minimization procedure may produce a notchless flat characteristic. One way of overcoming this difficulty is by imposing a rigid constraint such as

$$\Omega[y(0)] = F(y(0,\omega_0)) = |G(\omega_0)|^2 = 0 . \qquad (3.2.11)$$

We will use Improved First Order Gradient Technique for tackling this problem. We seek a $\delta u(x)$ that will give a specified improvement $d\phi$. From expression (3.2.10).[6]

$$d\phi = \sum_{\omega_i} dF(\omega_i) , \quad i = -m_1,\ldots,-1,0,1,\ldots m_2 .$$
$$(3.2.12)$$

The function $F(\omega_i)$ is the measurement of the error, or deviation from desired characteristics at frequency $\omega_i$. Thus, $dF(\omega_i)$ is a variation in error at $\omega_i$. Starting with equation (1.3.1) one can follow the development up to (1.3.22), obtaining

$$dF(\omega_i) = \int H_u(x,\omega_i)\delta u(x)dx . \qquad (3.2.13)$$

---

[6]The multiplying factor $\Delta$ will not in any way affect the minimization procedure and hence can be dropped at this point.

The expression (3.2.12) gives variation $d\phi$ as a weighted sum of the variations $dF(\omega_i)$; the weighting is uniform in this case. Apparently the summation over $\omega_i$ will suppress the information available in equations (3.2.13)' regarding the behavior of the frequency characteristics at each $\omega_i$. It is not necessary to loose this information. Instead of specifying $d\phi$ and letting the weighting factors distribute the correction over $\omega_i$, we could specify each $dF(\omega_i)$ separately. Thus, we will be specifying the entire contour of the improved frequency characteristics. This amounts to specifying $(m_1+m_2+1)dF(\omega_i)$ and having $(m_1+m_2+1)$ equations, as specified by (3.2.13), to be complied with at every iteration cycle. Equation (3.2.11) is one of these $(m_1+m_2+1)$ equations. As will be apparent later, we pay the price by having to solve $(m_1+m_2+1)$ simultaneous equations to obtain the required variation in control $\delta u$.

3.3     Method of Solution

The system being linear, its behavior at any frequency $\omega_i$, can be studied independently. Thus, we have $(m_1+m_2+1)$ independent system equations

$$\frac{d}{dx} y(x,\omega_i) = A(x,\omega_i)y(x,\omega_i) \quad , \qquad (3.3.1)$$

with boundary conditions

$$y^t[x=L,\omega_i] = [a,0,0,0] \quad . \qquad (3.3.2)$$

There are $(m_1+m_2+1)$ adjoint equations

$$\frac{d}{dx} \lambda(x,\omega_i) = -A^t(x,\omega_i)\lambda(x,\omega_i) \quad , \qquad (3.3.3)$$

with boundary conditions

$$\lambda^t(x=0,\omega_i) = -F_y\big|_{x=0} \quad . \qquad (3.3.4)$$

and $H(x,\omega_i)$ in (3.2.13) is given by

$$H(x,\omega_i) = \lambda^t(x,\omega_i)A(x,\omega_i)y(x,\omega_i) \quad . \qquad (3.3.5)$$

We seek $\delta u$ that satisfies $(m_1+m_2+1)$ equations specified by (3.2.13). The values of $dF(\omega_i)$ are chosen so as to improve the frequency response characteristics and

$$dF(\omega_0) = -F(\omega_0) \quad . $$

As in Sections 1.3 and 2.3, an accessory minimization problem

can be formulated with $\delta u$ as a control variable. We can write

the composite criterion functional

$$\psi = \frac{1}{2} \int \delta u^t W \delta u\, dx + \sum_i v_i [dF(\omega_i) - \int H_u(\omega_i) \delta u\, dx] \quad , \tag{3.3.6}$$

where $v_i$ are undefined Lagrange multipliers. The Euler Lagrange

Equations for this minimization problem yield

$$\delta u = W^{-1} [\sum_i v_i H_u(\omega_i)]$$

Define

$$\delta u_i^\cdot = W^{-1} H_u(\omega_i) \tag{3.3.7}$$

Hence

$$\delta u = \sum_i v_i \delta u_i \tag{3.3.8}$$

Thus, $\delta u$ is composed of $(m_1 + m_2 + 1)$ components $\delta u_i$ and corresponding stepsizes $v_i$. Substituting (3.3.8) into (3.2.13) we obtain

the simultaneous equations in $v_i$

$$dF(\omega_i) = \sum_{j=-m_1}^{m_2} v_j \int H(\omega_i) \delta u_j\, dx \tag{3.3.9}$$

For a nominal control vector u, $(m_1 + m_2 + 1)$ system and adjoint

equations can be solved and $H_u$ can be evaluated as defined by

(3.3.5). The shape factors "$\delta u_i$" can be evaluated from (3.3.7).

Thus, the equations (3.3.9) become a set of simultaneous algebraic

equations in $v_i$.

The iteration algorithm is similar to one presented in Part

A of Section 2.3. The details of the algorithm are covered in

the following section during the description of the computer flow chart.

The same difficulties of convergence as described in Part A of Section 2.4 are faced here. As soon as the control reaches the boundary values and $u+\delta u$ starts getting truncated, the improvement in criterion function does not correspond to the stipulated values of $dF(\omega_i)$. It becomes necessary to use the "shaping factor" or "the convergence factor" $W(x)$ other than the identity matrix. Let the estimate of variation in control after truncation be $\delta u'$ (see Fig. 2.10), then the convergence factor $W$ is chosen to be

$$
W^{-1} = \begin{bmatrix} |\delta r'| \left[ \frac{(r_M - r_m)}{L} x + r_m \right] & \\ & |\delta c'| \left[ c_M - \frac{(c_M - c_m)}{L} x \right] \end{bmatrix}
$$

(3.3.10)

With this definition of $W^{-1}$ a second estimate of $\delta u_i$ is obtained from (3.3.7) and hence a second estimate of $v_i$ from (3.3.9). Equation (3.3.9) yields new $\delta u$. The revised estimate of the control $u+\delta u$ is checked for bounds, truncated if necessary and accepted as an improved control. The program then starts a new iteration cycle.

3.4        Computer Program

The iterative solutions are obtained on the Hybrid Computer
using the improved first order gradient technique.

(i)   Analog Patching

The system proper is described by

$$\frac{d}{dz} y(z,\omega_i) = - A(z,\omega_i)y(z,\omega_i) \quad , \qquad (3.4.1)$$

with the boundary conditions

$$y^t(z=0,\omega_i) = [a,0,0,0] \quad , \qquad (3.4.2)$$

where

$$z = (L-x) \quad .$$

The adjoint system equations are

$$\frac{d}{dx} (x,\omega_i) = - A^t(x,\omega_i)\lambda(x,\omega_i) \quad . \qquad (3.4.3)$$

With the transformation[7]

$$\Upsilon(x,\omega_i) = B\lambda(x,\omega_i) \quad , \qquad (3.4.4)$$

where

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

---

[7] See equation (1.3.10).

the equation (3.4.3) reduces to

$$\frac{d}{dx} \gamma(x,\omega_i) = - A(x,\omega_i)\gamma(x,\omega_i) \quad . \tag{3.4.5}$$

The form of these equations is identical to (3.4.1). Thus, with a proper choice of variable coefficients and initial conditions, the same set of equations yield the solutions $y(z,\omega_i)$ and $\lambda(x,\omega_i)$.

The boundary conditions for $\lambda(x,\omega_i)$ are obtained from (3.3.4). We have already defined the error function

$$F(\omega_i) = \left| G_d(\omega_i^2) - G(\omega_i^2) \right|$$

$$= \left| G_e(\omega_i) \right|$$

We will need a set of concise definitions in order to keep the algebra straight. Thus, let us define for every $\omega_i$

$$DTR = (y_1(x=0)+y_3(x=0)R)^2 + (y_2(x=0)+y_4(x=0)R)^2 \quad ,$$

$$NTR = (a+y_3(x=0)R)^2 + (y_4(x=0)R)^2 \quad .$$

Then equation (3.3.4) gives the boundary conditions for $\lambda$ at $x=0$ and $\omega=\omega_i$

$$\lambda_1(0) = - 2 \operatorname{sgn}(G_e) \frac{NTR(y_1(0)+y_3(0)R)}{DTR^2} \quad ,$$

$$\lambda_2(0) = - 2 \operatorname{sgn}(G_e) \frac{NTR(y_2(0)+y_4(0)R)}{DTR^2} \quad ,$$

$$\lambda_3(0) = 2 \text{ sgn } (G_e) \frac{DTR(a+y_3(0)R) - NTR(y_1(0)+y_3(0)R)}{DTR^2} ,$$

$$\lambda_4(0) = 2 \text{ sgn } (G_e) \frac{DTR(y_4(0)R) - NTR(y_2(0)+y_4(0)R)}{DTR^2} .$$

$$(3.4.6)$$

The analog computer patching is given in Fig. 2.6.

(ii) Flow Diagram

The algorithm used is similar to one described in Part A of Section 2.4. The flow chart for the program is given in Fig. 3.5. The detailed explanation for the flow chart is as follows.

Block 1: Preparatory Steps.

The input output channels are reset; the data, such as the bounds on the control variables, etc. is read in. The initial profiles of the control variables are loaded into the memory.

Block 2: Solving the system equation on Hybrid Unit.

For each value of $\omega_i$ the elements of matrix $A(z,\omega_i)$ are evaluated. The initial conditions are set up as given by (3.4.2). The equations are integrated backwards in space on the analog computer. The system variables are stored in the memory of the digital computer. The quantities $|G(\omega_i^2)|$ and hence $F(\omega_i)$ are obtained from (3.2.9) and (3.2.10). In all $(m_1+m_2+1)$ sets of equations are solved -- one for each $\omega_i$. The values for $dF(\omega_i)$ are chosen so as to drive $F(\omega_i)$ towards zero. The rigid

```
┌─────────────────────────────────┐
│   1.   Initial Setting          │
└─────────────────────────────────┘
```

```
┌──────────┐   ┌─────────────────────────────┐   ┌──────────┐
│ Store    │◄──│ 2.  Solve system equations. │──►│ dF(ωᵢ)   │  Yes
│ y(ωᵢ)    │   │  Check dF(ωᵢ).   Set new    │   │  too     │  Exit
│          │   │        dF(ωᵢ)               │   │  small?  │
└──────────┘   └─────────────────────────────┘   └──────────┘
```

```
┌──────────┐   ┌─────────────────────────────┐        No
│ Store    │◄──│   3.   Solve adjoint        │◄──────
│ λ(ωᵢ)    │   │  equations for λ(ωᵢ)        │
└──────────┘   └─────────────────────────────┘
```

```
               ┌─────────────────────────────┐
               │  4.  Evaluate the first     │
               │  estimate of δuᵢ and        │
               │  vᵢ, hence δu.  Check       │
               │  u+δu for bounds            │
               └─────────────────────────────┘
```

```
               ┌─────────────────────────────┐
               │   5.  Obtain second         │
               │   estimate of δuᵢ           │
               │        and vᵢ               │
               └─────────────────────────────┘
```

```
               ┌─────────────────────────────┐
               │   6.  New δu and u+δu.      │
               │   Check for bounds          │
               └─────────────────────────────┘
```
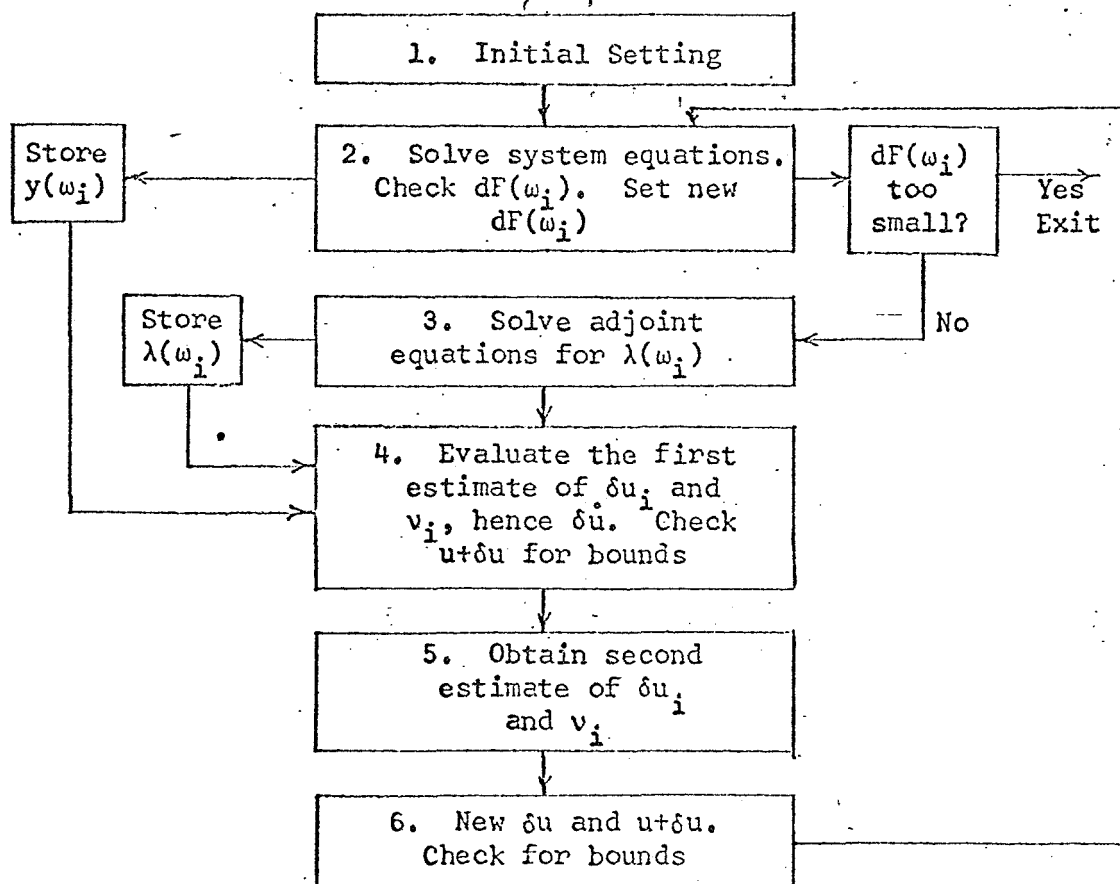
FIGURE 3.5

COMPUTING ALGORITHM FOR NOTCHED FILTER DESIGN USING
IMPROVED FIRST ORDER GRADIENT TECHNIQUE

constraint at null frequency is taken into account by setting

$dF(\omega_0) = -F(\omega_0)$.

Block 3. Solving the Adjoint Equations on Hybrid Unit.

For each value of $\omega_i$ the elements of matrix $A(x,\omega_i)$ are

evaluated. The initial conditions are set up as given by

(3.4.6). The equations are integrated forward in space on the

analog computer. The adjoint variables are stored in the memory

of the digital computer. The same analog program that is used

for the adjoint system with the transformation of variables

given by (3.4.4). In all $(m_1+m_2+1)$ sets of equations are solved

for each $\omega_i$.

Block 4. First Estimate of $\delta u$.

Assuming W to be an identity matrix $\delta u_i$ is obtained from

(3.3.7). Simultaneous equations (3.3.9) are solved to obtain

$v_i$. The first estimate of $\delta u$ is obtained from (3.3.8). The

new control $u+\delta u$ is checked for bounds and truncated if

necessary, thus obtaining $\delta u'$ (see Fig. 2.10) as an allowable

variation in u.

Block 5: Second Estimate of $\delta u$.

The matrix $W^{-1}$ is evaluated from (3.3.10). With this value

of $W^{-1}$, $\delta u_i$, $v_i$, and $\delta u$ are evaluated from (3.3.7), (3.3.9), and

(3.3.8) respectively. This is a second estimate of $\delta u$. The

program uses the first estimate of $\delta u$ until the control reaches

one of the bounds. When the control reaches a limiting value

the estimates of $dF(\omega_i)$ do not correspond very well to the actual

improvement effected by the updated -- and truncated -- control.
At this point the program begins to obtain a second estimate --
using matrix $W^{-1}$ from (3.3.10).

The u+δu obtained from this second estimate of .δu is
checked for bounds and truncated if necessary. The program now
braches back to Block 2 for the next iteration loop.

## 3.5    Numerical Solutions

The numerical solutions were obtained on a Hybrid Computer. As a particular example, the ratio of $r_M/r_m$ and $c_M/c_m$ is chosen[8] to be 10. The limiting values are chosen to be

$$r_M = \omega_0 c_M = .8$$

$$r_m = \omega_0 c_m = .08 \qquad (3.5.1)$$

where $\omega_0$ is the null frequency of the notched filter.

To start the iterative procedure the nominal distributions are assumed to be

$$r(x) = \omega_0 c(x) = 0.352$$

$$\omega_0 \ell(x) = 0 \qquad (3.5.2)$$

and

$$R = .198$$

The length of the line L is held constant at 10 units. These values are obtained from the results reported by Fuller and Castro [22].

We have considered only three points in the frequency domain, $\frac{1}{2}\omega_0$, $\omega_0$, and $2\omega_0$. Thus $\omega_i$ takes only three values

---

[8]See Section 2.5.

$$\omega_{-1} = \frac{1}{2} \omega_0 \quad,$$

$$\omega_0 = \omega_0 \quad,$$

$$\omega_1 = 2\omega_0 \quad.$$

It is observed that for a given variation in $\delta u$ the variation $dF(\omega_i)$ moves in the same direction in the frequency range $\omega_0 < \omega_i \leq 10 \, \omega_0$. The same holds good for $\omega_0 > \omega_i \geq \frac{1}{10} \, \omega_0$. Thus, $\frac{1}{2} \omega_0$, $\omega_0$, and $2\omega_0$ give a good representation of the frequency response characteristics in the range $\frac{1}{10} \omega_0$ to $10 \, \omega_0$.

The final "optimal" distributions are given in Fig. 3.6. The distributed inductance $\ell(x)$ is assumed to be zero. During the iterations $H_u$ drops down by a factor of about 500 indicating that the final distributions are very close to the "optimal". Figure 3.8 gives the frequency response characteristics for (i) initial distributions as given by (3.5.2) and (ii) the "final" distributions as in Fig. 3.6.

Figure 3.7 gives the initial and final distributions for a case where $\omega_0 \ell(x) = .04$. The distributed inductance is assumed to be constant and non-controllable.
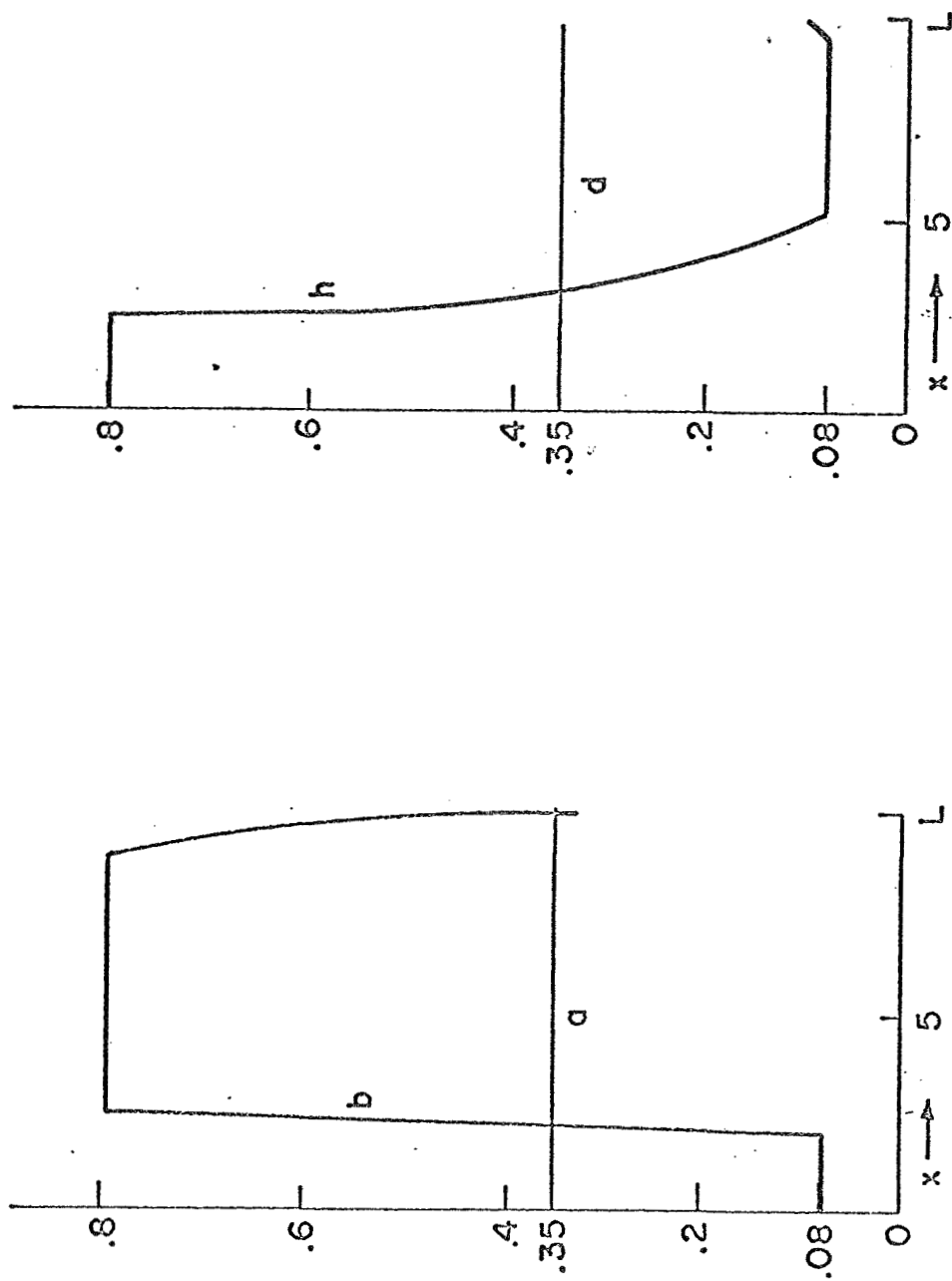
FIGURE 3.6

THE INITIAL AND OPTIMAL DISTRIBUTIONS
FOR THE NOTCHED FILTER

a — $r(x)$, Initial Distribution
b — $r(x)$, Final Distribution

d — $\omega_0 C(x)$, Initial Distribution
h — $\omega_0 C(x)$, Final Distribution

$$L=10, \quad \omega_0 \ell(x)=0$$

FIGURE 3.7

a – r(x), Initial Distribution          d – $\omega_0 C(x)$, Initial Distribution
b – r(x), Final Distribution            h – $\omega_0 C(x)$, Final Distribution

L=10 , $\omega_0 \ell(x)=0.04$

THE INITIAL AND OPTIMAL DISTRIBUTIONS
FOR THE NOTCHED FILTER
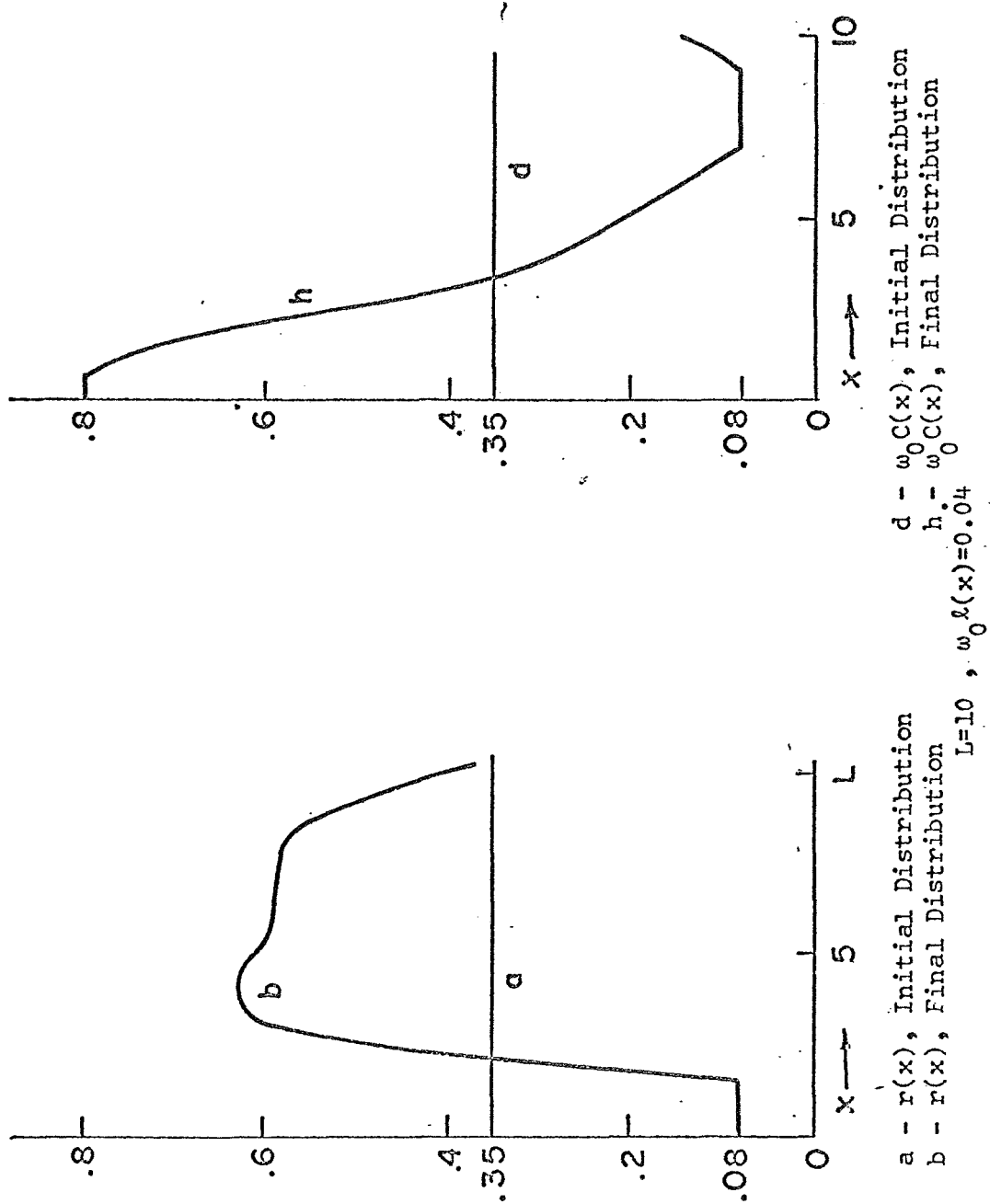
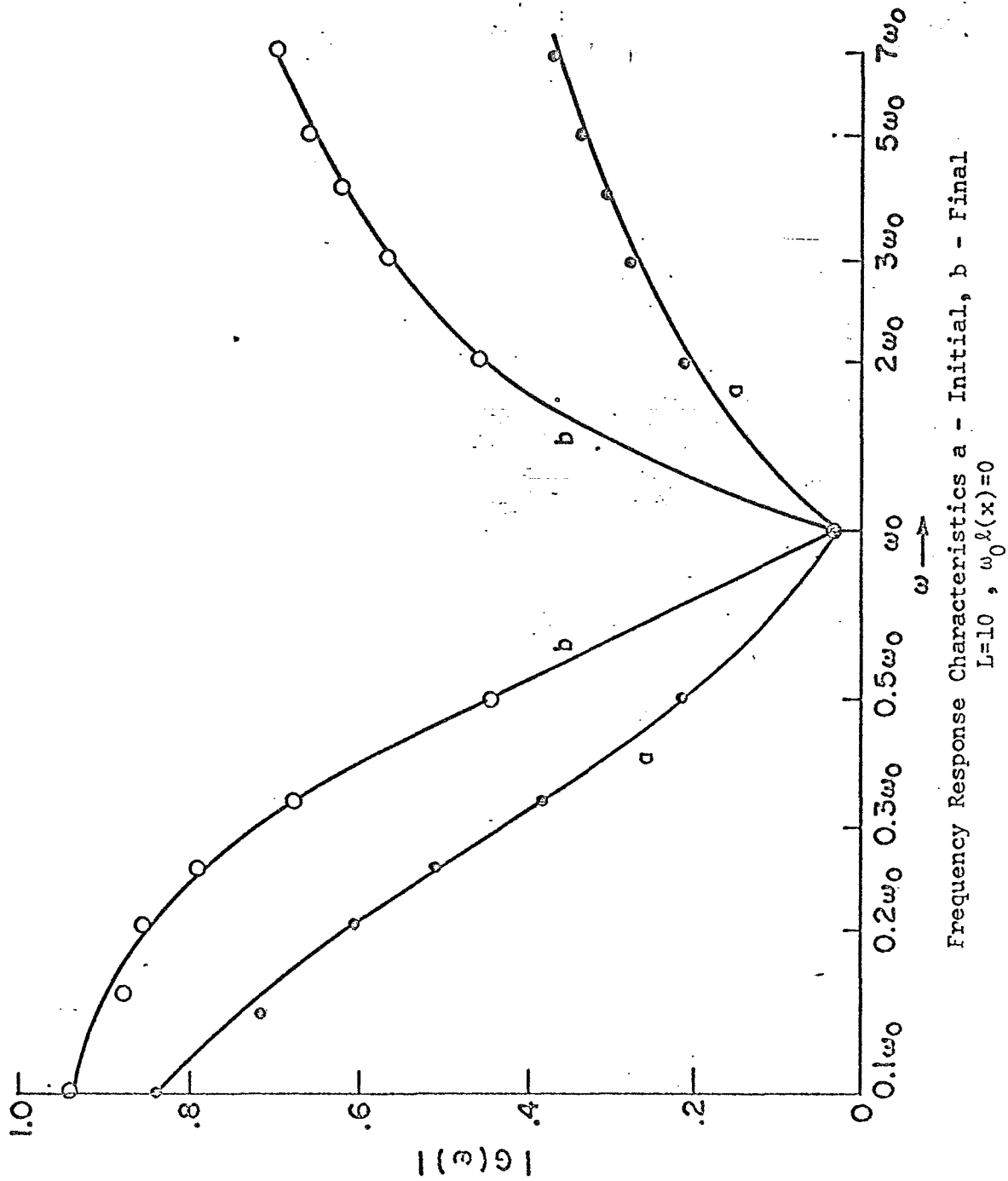Frequency Response Characteristics a - Initial, b - Final
$L=10$ , $\omega_0 \ell(x)=0$

FIGURE 3.8

THE FREQUENCY RESPONSE CHARACTERISTICS OF THE NOTCHED FILTER FOR INITIAL NONOPTIMAL
AND THE OPTIMAL RESISTANCE AND CAPACITANCE DISTRIBUTIONS

Errors and Limitations

A. Scale and Range

The analog computer is a 10 volt machine. The DAC is a 10 volt unit with 14 bits plus a sign bit and the ADC is a 10 volt unit with 13 bits plus a sign bit. Thus, the lowest voltage level that the setup can handle is about 2 mv, as decided upon by the ADC. Any voltage level below 2 mv is interpreted as a zero by the ADC and the voltage levels above 10 volts are either rejected by the converters or cause saturation of the amplifiers. Thus, the dynamic range of the setup is $5 \times 10^3$.

B. Noise

(i) Random Noise -- The individual component of the system has a specified noise level as given below.

ADC - the noise level is $\pm 1$ bit, equivalent to about $\pm 2$ mv.

DAC - the noise level is negligible as compared to that of the ADC and analog computer.

Analog Computer - the nonlinear multipliers have the highest noise level. It is specified to be $\pm 3$ mv. However, when the transmission line equations were integrated a number of times using the entire Hybrid setup, for the same distributions r(x), c(x) and $\ell(x)$ the end point values of the voltage $V_1(x)$ were found to be repeatable within 20 mv.

(ii) Quantization Noise -- The ADC while reading the results from the analog computer quantizes them. The random

noise is superposed on top of this quantized signal. In the
algorithms these readings are operated upon and amplified --
especially during the last part of the iteration -- several
times. Thus, 2 mv quantization step and about 6 mv noise can
cause a noise level in the range of a hundred mv. Figures 2.12
through 2.22 are the smoothed out versions of the computer
output. Figure 2.11 is that of an original computer output.

The noise problem becomes more serious with the complicated
algorithms involving large numbers of algebraic operations.
For this reason, the algorithm should be as simple as possible.

### C. Limitations of the Method

$H_u$ is a smoothly varying function. Thus, every variation
in the control has a continuous first derivative in the open
region. If the optimal distribution has a discontinuous first
derivative and the initial estimate does not, the solution will
not converge on to the optimal. Also if the initial guess has
a discontinuous first derivative, we can never get rid of this
discontinuity in the open region. In the present case, the
uniform, ramp, exponential distributions all converged to the
same distribution. However, when the initial guess was a bang
bang type of distribution, the final distribution retained the
discontinuities in the first derivative.

# CHAPTER 5
## CONCLUSIONS

As stated in the Introduction, the aim of the present study was to develop a technique for the synthesis of the optimal distributed parameter systems, such as a transmission line, where the parameters are bounded. The problem is formulated as an optimal control problem with the parameter distributions as the control variables. The gradient technique was preferred to all other approaches because of its property of stepwise improvement in the criterion function. The hybrid computational technique seems to be best suited for the gradient method (see Section 2.5).

In obtaining the optimal distributions of the parameters $r(x)$ and $c(x)$ using the first order gradient technique, two types of convergence problems were encountered.

(1) Sensitivity: As described in Part A of Section 2.4, the ratio of the components of the sensitivity function $H_u$ and the total desired variation $u_{opt}-u$ may vary considerably over the range of $x$. In the case of a one-dimensional control vector, this ratio is a good indicator of the degree of convergence with respect to the number of iterations required to reach "sufficiently" close to the optimum. A constant ratio can be compensated by a proper choice of the factor $W$. This presumes some knowledge regarding the system behavior and the nature of the optimum distribution.

(2) Truncation: The bounds on the control variables require that the variations in the control variables be restrained wherever the control tries to cross the bounds. This gives rise to the convergence problem described in Section 2.4, Part A. Again, a proper choice of W matrix can eliminate this problem. The first estimate of the variation can be used to obtain the weighting factors in the form of $W^{-1}$. These factors can be used to obtain a second estimate of the required variation.

With the bounds on resistance and capacitance decided upon by the fabrication limitations, and the length prespecified, the optimum 180 degree phase shift network with minimum attenuation turns out to have distributions of r and c that have limiting values with the singular switching curves.

The attenuation of unity, as projected by Johnson and calculated from Edson's results is not realizable due to the physical limitations.

The optimum attenuation is not far better than what can be achieved by exponential distributions given a free choice of length.

With the bounds on resistance and capacitance decided upon by the fabrication limitations, and the length prespecified, the optimum notched filter with the configuration as shown in Fig. 3.3 .turns out to have distributions of r and c that have limiting values with the singular switching curves.

$H_u$ is a smoothly varying function. Thus, every variation
in the control has a continuous first derivative in the open
region. If either the optimal distribution or the initial guess
has a discontinuous first derivative in the open region, one
should expect problems of convergence. In the cases presented
above, the same final distributions were obtained whether one
assumes an initial uniform distribution or an initial ramp distri-
bution. However, when the initial guess was a bang bang type of
distribution, the final distribution retained the discontinuities
in the first derivative.

We have shown here that it is possible to obtain a solution
to a "singular" optimization problem by using the Improved
Gradient Technique developed here. It is applicable to a large
spectrum of problems in the transmission processes.

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## Hybrid Computer System

This is a combination of the analog and digital computers.
We have EA1680 analog computer and IBM7700 digital computer
with input-output subchannels for the transfer of the informa-
tion. In order to transform this setup into a Hybrid unit, we
designed and built the interface. Figure A.1 shows the flow
diagram for the Hybrid unit.

(i) Digital computer: The digital computer contains the
multiplexor channel, channel B. It permits the attachment of
different data acquisition and data distribution devices to the
processor of the digital computer. The input subchannels of
channel B are capable of recording the logic levels -- true or
false -- of the incoming lines and the output subchannels can
send the desired logic levels on the output lines. The opera-
tion of channel B is controlled by the central processor unit.

(ii) Interface: The interface provides the medium of
communication between the analog computer and digital computer.
It is essentially a translator unit. The function of the various
sections of the interface are described below.

Operation Control of the Analog Computer: The operation of
the analog computer is controlled by the coded logic signals sent
from the digital computer. The interface converts the input
logic levels and also generates the clock pulses required for
certain operations.

FIGURE A.1

HYBRID COMPUTER FLOW CHART
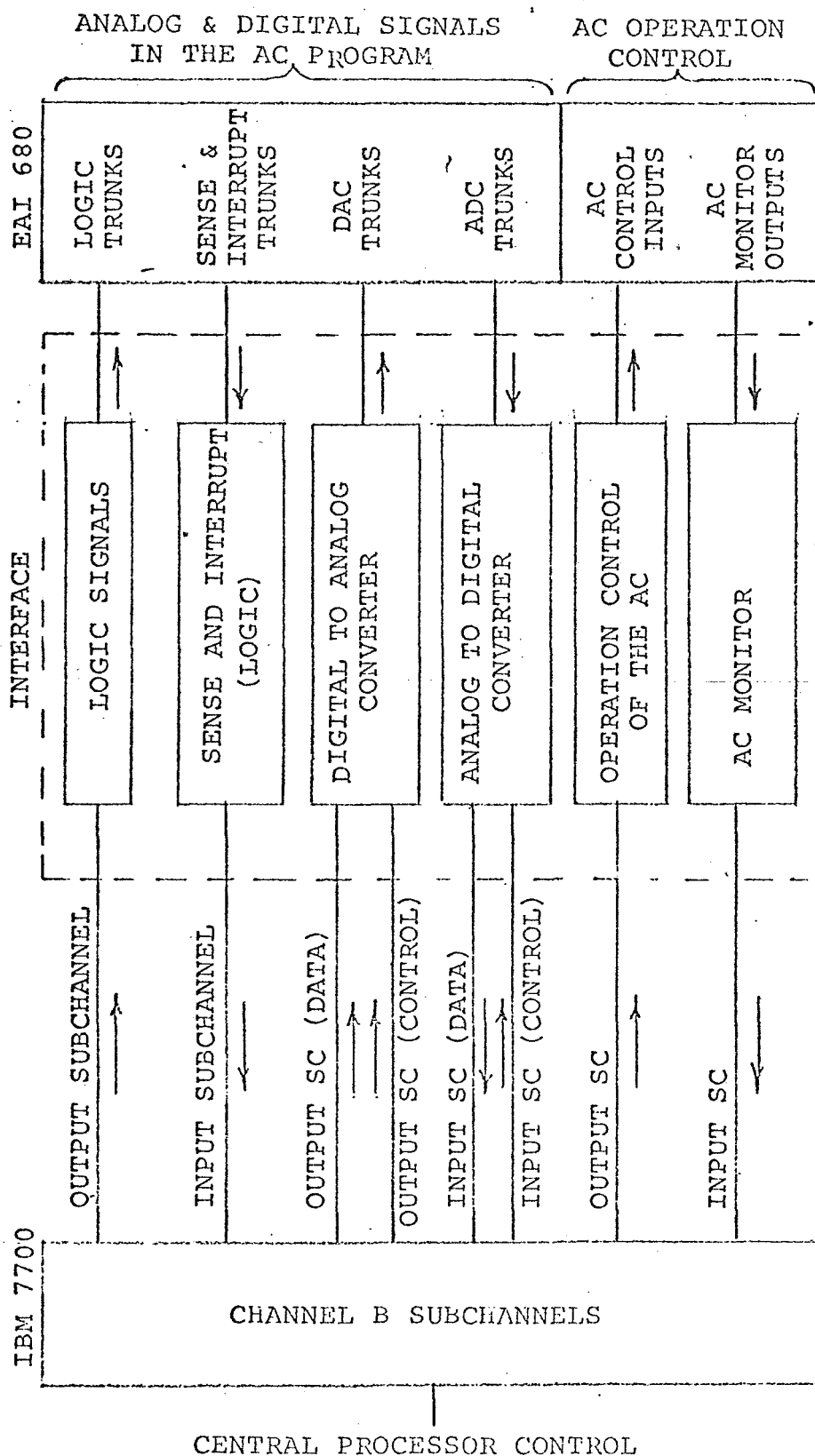
The arrows indicate the direction of the flow of the information.

The operations controlled are as follows:

(1)  Operate (Integrate), Hold, Initial condition, etc.

(2)  Analog component selction for readout or potset;
     e.g., Amplifier, Trunk, Pot, etc.

(3)  Time constant selection, e.g., Seconds, Milliseconds,
     etc.

(4)  Digital mode selection, e.g., Set, Clear (Registers,
     Counter), etc.

(5)  Digital clock rate selection

(6)  Selecting the address of the analog component

(7)  Setting a pot coefficient

Analog Computer Monitor:  The coded logic signals coming from
the monitor of the analog computer are transmitted to the digital
computer.  The digital computer compares the control order with
the monitor signal to find out whether the execution is proper.

Logic Signals:  Certain decisions made by the digital computer
regarding the status of the program under execution are trans-
mitted through interface to the logic trunks.  These signals can
be used to effect a change in the analog computer program.

Sense and Interrupt:  The status of the analog computer
program such as a comparator output is conveyed to the interface
on the sense lines.  The interface in turn transmits the message
to the digital computer.  The interrupt lines are used for convey-
ing the undesirable status of operation such as overload.  The

analog computer is programmed to interrupt the operation under such conditions.

Digital to Analog Converter: This is an eight channel serial input, parallel output unit. The control signal from the analog computer initiates the conversion of the digital data on the input lines from the digital computer into the analog signal. The analog signal appears on the channel selected by the control word from the digital computer. The output channels are connected to the DAC trunks on the analog computer.

Analog to Digital Converter: This is a 24 channel parallel input serial output unit. It receives the analog input from the ADC trunks. The control word from the digital computer selects the channel and initiates the conversion. The digital output is transmitted to the digital computer.

(iii) Analog Computer: The analog computer can be divided into three sections.

Analog Section: It consists of the analog components such as integrators, summing amplifiers, track and store amplifiers, etc. ADC trunks receive the inputs from this section and DAC trunks supply the analog signals to this section.

Logic Section: This section contains the logic elements such as gates, counters, registers, along with the clock outputs and control inputs for certain analog components. The sense and interrupt trunks receive the inputs from this section. The logic trunks appear in this section.

Operation Control: This section controls the operation of both the analog and logic sections. It controls all of the operations listed under "Operation Control of the AC" in the description of the interface. It receives the coded control word, either from pushbuttons or from the interface. It also generates the monitor signals.

## Hybrid Operations

The two important links in the hybrid setup are the DAC and the ADC.

DAC: The output subchannel of the digital computer transmits the digitized value of the variable. The load command from the digital computer loads the word into the registers of the DAC. However, unless the DAC channel receives the enable command, the analog output does not appear at the output terminal of the DAC. The previous value is retained at the output until a new enable command is received.

ADC: The digital computer selects the ADC channel by controlling the multiplexor switches. The conversion of the analog signal on this preselected channel is initiated by the start pulse. On completion of the conversion a pulse is sent to the input subchannel of the digital computer. On receiving this pulse the input subchannel registers the digital output of the ADC. This is subsequently transferred to the memory of the digital computer.

Setting up initial conditions and static test:

The operation control subroutine sets the analog computer in the "set pot" mode. The proper address word selects the desired servo-controlled pot. The value register is loaded and the servo start pulse transmitted from the digital computer. The monitor subroutine checks if the proper pot has been selected and the operation completed. Thus, the initial condition -IC- is established with the help of servoset pots.

The analog computer is then driven into the IC mode and outputs of amplifiers are read on the ADC. This gives the static test.

Integration routine: A subchannel of the digital computer is used for starting and terminating the integration operation. Selection of the counter SC turns trunk "00" (Fig. A.2) on and the analog computer goes into "operate" mode thus starting integration. At the same time, the analog computer counter starts counting analog computer clock pulses and gives the output as in Fig. A.2. The monostable multivibrator (Fig. A.3) generates a pulse every 1000 sec. which generates a digital computer interrupt. The digital computer counts the number of such interrupts. As soon as the digital computer counts a specified number of pulses, it deselects the subchannel terminating the integration operation and driving the analog computer into the IC mode. The pulse from the monostable multivibrator also starts the conversion and enables the DAC channels.

Before the start of integration:

(i)   The analog computer counter is reset;

(ii)  The analog computer clock mode is selected (such as
      10 kc, 100 kc, 1000 kc);

(iii) The analog computer time constant is selected (such
      as seconds, milliseconds, etc.);

(iv)  The values of DAC functions for the second interval
      are loaded.

(For the oscillator problem, the clock mode was 1000 kc and
the time constant was 0.1 sec.)

Now the integration is started by selecting the counter SC.
Figure A.4 describes the flow of events.

As the first counter pulse comes in, it enables all the DAC
channels. Thus, values of all the coefficents for the second
interval are made available. All the track and store amplifiers
go into store mode thus preserving the values at the instant of
the counter pulse. The digital computer now selects and reads
the ADC channels one by one. This is followed by serial loading
of DAC channels with the values for the next interval. This
completes the operations for one interval and the digital computer
waits for next counter pulse. The process repeats until the
counter SC is deselected.
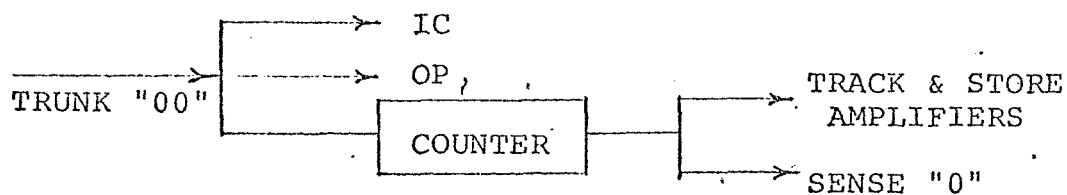
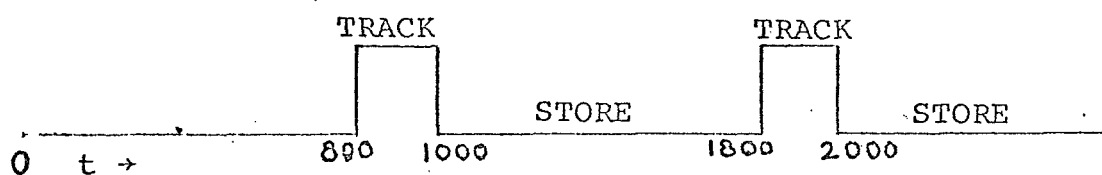LOGIC PATCHING OF THE ANALOG COMPUTER
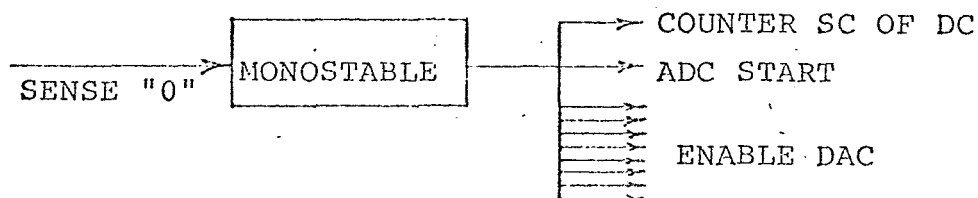


FIGURE A.2

COUNTER OUTPUT OF THE ANALOG COMPUTER
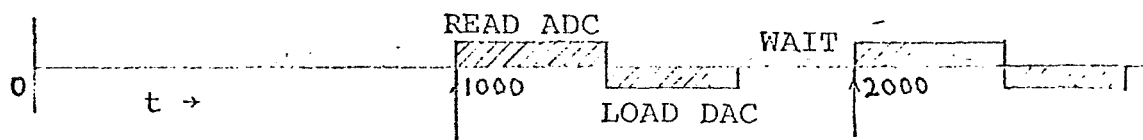


FIGURE A.3

MONOSTABLE MULTIVIBRATOR OUTPUTS



FIGURE A.4

THE OPERATIONS CONTROLLED BY SENSE "0" PULSE

## Truncation and Convergence

For a system represented by

$$\frac{d}{dx} y = f(y,u,x) \quad,$$

with criterion function $\phi(y(0),y(L))$, the functional relationship between a variation in $\phi$ and the variation '$\delta u$' in control u is obtained as (See equation (2.3.31))

$$d\phi = \int_0^L H_u \delta u\, dx \quad. \qquad\qquad (B.1)$$

Let us assume that $\phi$ is to be maximized. In the gradient technique the hope that the iterations would converge is based on obtaining a positive $d\phi$ as a result of every iteration cycle. Thus, we can stipulate three necessary conditions for $\delta u(x)$,

(i)  Sgn $\delta u(x) = $ Sgn $H_u(x)$, for a finite length, and

$\delta u(x) = 0$ for the rest of x.  This assures $d\phi \geq 0$,  (B.2)

(ii)  $u_m \leq u(x) + \delta u(x) \leq u_M$ ,

(iii)  $\int_0^L < \delta u(x), \delta u(x) > dx << 1$ .

This assures that the variation $\delta u(x)$ is small enough to justify the first order approximations made in the derivation of (2.3.31).

Let us define

$$u_m - u(x) = \delta u_m(x)$$

and

$$u_M - u(x) = \delta u_M(x) \quad.$$

The bounds on $\delta u(x)$ can now be specified as

$$\delta u_m \leq \delta u(x) \leq \delta u_M \quad .$$

Since $u(x)$ is an admissible control vector

$$u_m \leq u(x) \leq u_M \quad .$$

Hence

$$\delta u_m(x) < 0 \quad ,$$

and

$$\delta u_M(x) > 0 \quad .$$

Let $\delta u(x)$ be any function that satisfies the first and the last condition stated in (B.2). (See Fig. B.1). The function $\delta u$ can be expressed as a sum of a function $\delta u_p$ and $\delta u_n$ such that

$$\delta u_p(x) \geq 0 \quad ,$$

$$\delta u_n(x) \leq 0 \quad ,$$

and

$$\delta u = \delta u_p + \delta u_n \quad .$$

The variations $\delta u_p$ and $\delta u_n$ also satisfy the first and the last conditions stated in (B.2). The functions $\delta u_p(x)$ and $\delta u_n(x)$ can be further divided so that

$$\delta u_p(x) = \delta u_{pa}(x) + \delta u_{pt}(x) \quad ,$$
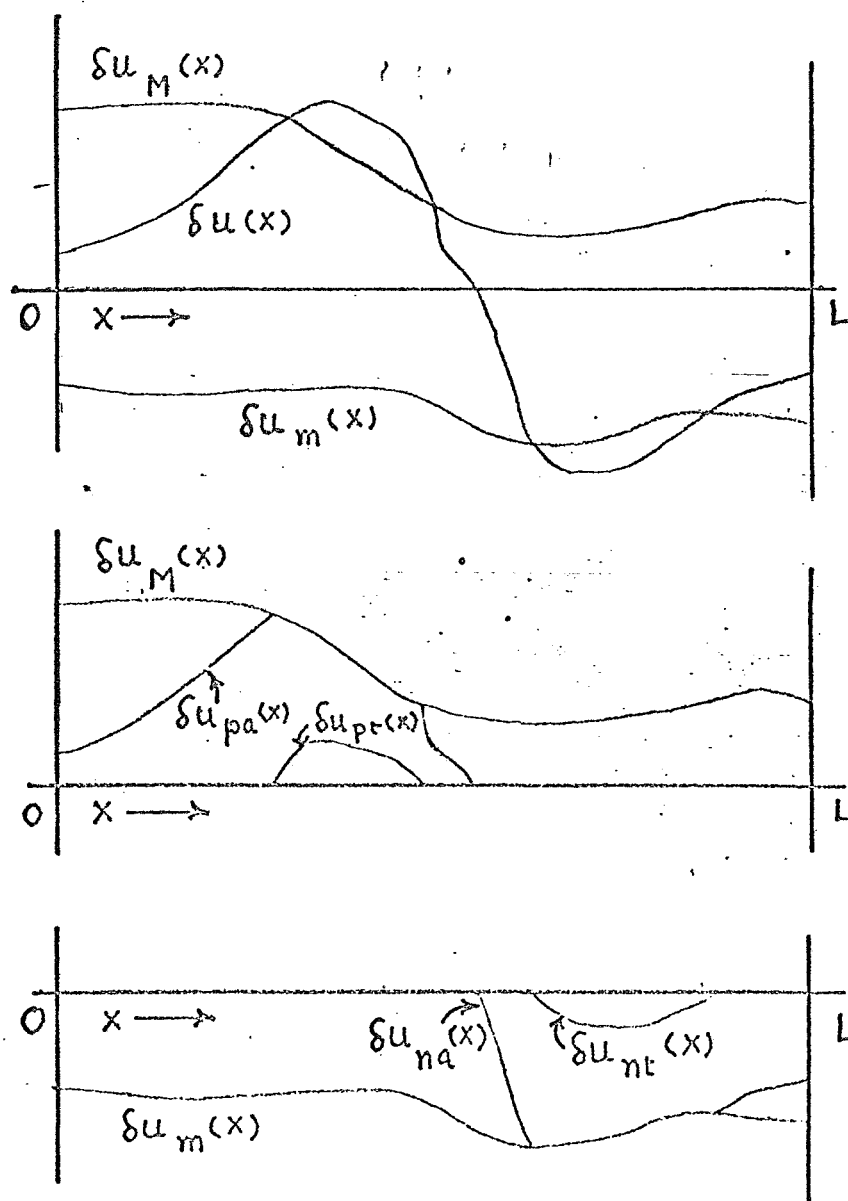
$$\delta u_n(x) = \delta u_{na}(x) + \delta u_{nt}(x) \quad ,$$

FIGURE B.1

TRUNCATION OF THE CONTROL VARIABLE
AT THE BOUNDS

where $\delta u_{pt}(x)$ and $\delta u_{nt}(x)$ are the truncated sections of $\delta u_p$ and $\delta u_n$ respectively.

We have

$$\text{Sgn } \delta u_p(x) = \text{Sgn } \delta u_{pa}(x)$$

and

$$\text{Sgn } \delta u_n(x) = \text{Sgn } \delta u_{na}(x)$$

Thus $\delta u_{pa}(x)$ and $\delta u_{na}(x)$ satisfies the first and the last condition stated in (B.2). They also satisfy the second condition (See Fig. B.1).

The same is true about $\delta u_a$ where

$$\delta u_a(x) = \delta u_{pa}(x) + \delta u_{na}(x) \qquad (B.3)$$

The function $\delta u_a(x)$ is a truncated part of $\delta u(x)$. Hence the truncation does not violate the conditions for convergence of the gradient method.

However, with more than one target function, such as $\phi$ and $\Omega$, $\delta u$ is composed of more than one component such as

$$\delta u = \nu^\phi \delta u^\phi + \nu^\Omega \delta u^\Omega \qquad (B.4)$$

and the functional relationship is (See (3.43)).

$$d\phi = \nu^\phi \int_0^L H_u^\phi \delta u^\phi dx + \nu^\Omega \int_0^L H_u^\phi \delta u^\Omega dx \qquad (B.5)$$

In such a case $\delta u^\Omega$ affects $d\phi$ (and $\delta u^\phi$ affect $d\Omega$). The condition (i) holds true for the first term on the R.H.S. of equation (B.5). However, the second term does not necessarily

satisfy the condition (i). Besides, $\delta u^\phi$ and $\delta u^\Omega$ are not truncated separately. The truncation of $\delta u$ does not provide any information as to how the truncation affects the components $\delta u^\phi$ and $\delta u^\Omega$. Thus the argument about convergence breaks down.

It is observed during the numerical calculations on computer that before the control distributions reach the limiting values, the first order gradient technique (using first estimate of $\delta u$) yields improvement in both $\phi$ and $\Omega$ simultaneously. However, once the control variables reach the boundary only one of the two improves and the other starts deteriorating. Thus, a simultaneous convergence breaks down.